THE VARIETY GENERATED BY TOURNAMENTS

By

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Dedicated to my loving wife, Adrienn,

who makes me want to be the best I can.

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Most of the material presented in this dissertation is the joint work of J. Ježek, P. Marković, R. McKenzie and myself. Our collaboration started when Petar and I were first year graduate students. We quickly got our initial results (Chapter III), and were exhilarated by our progress. This was the time when R. McKenzie made his conjecture about subdirectly irreducible algebras, which was later reduced by J. Ježek to strongly connected algebras (Chapter IV). Then came two laborious years, but we were unable to prove or disprove the conjecture. Looking back to the multitude of lemmas, many of which are surpassed now, we can find the precursors of three ideas: triangular subgraphs of tournaments from R. McKenzie, endomorphisms of free algebras from J. Ježek, and blow-up compositions of tournaments from P. Marković. These ideas helped me to prove the conjecture, my chief original contribution (Chapter V), and most of its consequences (Chapter VI). I must thank my collaborators for their continuing intellectual stimulation.

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CHAPTER I

INTRODUCTION

The program of systematic study of the algebraic properties of graphs, and relations in general, was carried out by K. Čulík, G. Sabidussi, Z. Hedrlín and A. Pultr. This approach led undoubtedly to success in applications of algebra and graph theory to various branches of mathematics. As part of this effort, V. Müller, J. Nešetřil and J. Pelant undertook the study of tournaments—a class of directed graphs that are basically the same as algebras of a certain kind (see [20]). By a tournament we mean a directed graph $\langle T; \rightarrow \rangle$ with all loops, such that whenever x and y are two distinct elements of T, then precisely one of the two cases, either $x \rightarrow y$ or $y \rightarrow x$, takes place. Already in 1965, Z. Hedrlín observed that a tournament $\langle T; \rightarrow \rangle$ can be made into a groupoid $\langle T; \cdot \rangle$, an algebra with a single binary operation, by defining $xy = yx = x$ if and only if $x \rightarrow y$. This correspondence between the class of all tournaments and the class of all commutative groupoids $\langle T; \cdot \rangle$ satisfying $xy \in \{x, y\}$ for all $x, y \in T$ is clearly a bijection. A moment of thought is enough to check that the graph homomorphisms and the algebraic homomorphisms are also in one-to-one correspondence and actually coincide. This makes it possible to identify tournaments with their corresponding groupoids and employ algebraic methods for their investigation.

Simple tournaments were studied in a different context by P. Erdős, E. Fried, A. Hajnal, E. C. Milner and J. W. Moon in [3], [4] and [18]. One of their main theorems states that (with the exception of chains with an odd number of elements) every tournament can be extended to a simple tournament by adding a single vertex. In [20], V. Müller, J. Nešetřil and J. Pelant characterize all finite lattices, called admissible lattices, that are isomorphic to the congruence lattice of a tournament, and sharpen a result of J. W. Moon [19] on the automorphism groups of tournaments.

Furthermore, they show that given an admissible lattice $\mathbf{L}$, an odd group $\mathbf{G}$, and a tournament $\mathbf{T}$, there exists a tournament whose congruence lattice is isomorphic to $\mathbf{L}$, whose automorphism group is isomorphic to $\mathbf{G}$, and that has a subtournament isomorphic to $\mathbf{T}$. This proves that, for tournaments, the congruence lattice and the group of automorphisms are independent.

One can easily check that tournaments satisfy, for example, the following equations.

(1) $xx = x$

(2) $xy = yx$

(3) $(xy)x = xy$

(4) $((xy)(xz))((xy)(yz)) = (xy)z$

On the other hand, the associative law is not satisfied, which can be verified in the three element cycle. In order to avoid too many parentheses, we adopt the following convention: $a_0a_1 \ldots a_{n-1}$ stands for $(((a_0a_1)a_2)\ldots)a_{n-1}$, and $a \cdot bc$ stands for $a(bc)$. Also, for example, $ab \cdot cd \cdot ef = ((ab)(cd))(ef)$.

It is natural to ask whether a list of equations like the one above is complete, in the sense that any equation satisfied by all tournaments would be derivable from the equations in the list. To answer this and similar questions, one needs to investigate not only tournaments in isolation, but the variety $\mathcal{T}$ of groupoids generated by tournaments. This leads outside the realm of graphs, as there are algebras in $\mathcal{T}$ which are not tournaments. Nonetheless, we can define a directed graph $\langle A; \rightarrow \rangle$ on each algebra $\mathbf{A} \in \mathcal{T}$ by writing $x \rightarrow y$ if and only if $xy = x$. For example, the direct square of the two element tournament is a semilattice, but not a tournament, because $ab \notin \{a, b\}$ (see Fig .1). We call pairs x, y of elements incomparable if $xy \notin \{x, y\}$.

In [2], [14] and [13], it has been proved that the variety $\mathcal{T}$ generated by tournaments is locally finite, non-finitely based, inherently non-finitely generated, and

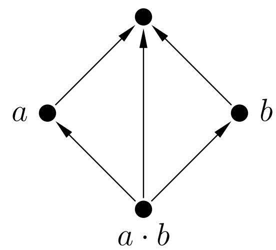

Figure 1: The direct square of the two element tournament

congruence meet-semidistributive. In the study of $\mathcal{T}$, R. McKenzie conjectured that all subdirectly irreducible members of $\mathcal{T}$ are tournaments (see [14]). Several partial results were obtained in this direction by J. Ježek, P. Marković, M. Maróti and R. McKenzie in [13], [11] and [15], but a proof of this conjecture remained out of reach. The main result of this dissertation establishes the truth of this conjecture. As the properties of subdirectly irreducible members of any variety greatly influence the properties of the variety itself, it was not very surprising that we found numerous consequences of this result. We prove that every finitely generated subvariety of $\mathcal{T}$ has a finite residual bound and is finitely based. The lattice of subvarieties of $\mathcal{T}$ is distributive, and we can describe the partially ordered set of join-irreducible members of this lattice. Finally, we give a representation theorem for all finite subdirectly irreducible members of $\mathcal{T}$ modulo simple tournaments.

Tournaments can be identified with algebras in two different ways. The approach to consider them as groupoids was taken, for example, in [9], [13], [14], [20] and in the present dissertation. Alternatively, tournaments can be also identified with algebras with two binary operations $x \cdot y$ and $x + y$, where $x \cdot y$ is defined as above and $x + y = x + y = y$ if and only if $x \rightarrow y$. This approach was taken, for example, in [5] and [6]. For tournaments themselves the difference is not significant. But, if we want to consider the variety generated by tournaments, we get different results for the two cases. For example, in the case of two binary operations, the variety generated by tournaments is contained in the variety of weakly associative lattices, and hence is congruence distributive (see [5]), in contrast to the fact that $\mathcal{T}$ is only congruence meet-semidistributive.

In the next chapter we review the basics of universal algebra that is essential for the understaning of the material in later chapters. Chapters III and IV collect some of the results published in [13], [14] and [11]. In Chapter V we present our main result, the proof of R. McKenzie's conjecture. Finally, in Chapter VI, we give some consequences of this result.

CHAPTER II

UNIVERSAL ALGEBRAIC BACKGROUND

The reader is referred to the excellent books [1] and [17] on universal algebra and equational theory. Some of the facts that are essential for the understanding of our results are recalled here.

We will assume the familiarity with the most basic notions of set theory. We use upper-case Latin letters to denote sets, and lower-case letters to denote elements of sets and integers. Let n be a non-negative integer. By an n-ary operation on a set A we mean a mapping of A^n to A, and by an n-ary relation on a set A, a subset of A^n. We call a nonvoid set A endowed with an indexed set $F = \{\, f_i^{\mathbf{A}} : i \in I \,\}$ of operations, an indexed set $R = \{\, r_j^{\mathbf{A}} : j \in J \,\}$ of relations, or both, an algebra $\langle A; F\rangle$, a relational structure $\langle A; R\rangle$, or an algebraic structure $\langle A; F, R\rangle$, respectively. This concept includes groups, rings, graphs (with no multiple edges), lattices, partially ordered sets and many other algebraic systems of interest in mathematics. We use boldface, upper-case letters to denote algebras and structures, and normal-font, lower-case letters or special symbols to denote operations and relations. For operations $f^{\mathbf{A}} \in F$ and relations $r^{\mathbf{A}} \in R$ we usually write f and r, respectively, if the algebra or structure $\mathbf{A}$ in question is known, and no confusion is likely to arise. We call an algebra $\mathbf{A} = \langle A; \cdot\rangle$ with a single binary operation $\cdot$ a groupoid, and a relational structure $\mathbf{B} = \langle B; \to\rangle$ with a single binary relation $\to$ a directed graph (with no multiple edges). For binary operations and relations denoted by special symbols we use infix notation.

For a set A define $\mathrm{id}_A = \{\, \langle a, a\rangle : a \in A \,\}$. We call a binary relation $\varrho \subseteq A^2$ reflexive if $\mathrm{id}_A \subseteq \varrho$, symmetric if $\langle a, b\rangle \in \varrho$ whenever $\langle b, a\rangle \in \varrho$, antisymmetric if $\langle a, b\rangle \notin \varrho$ whenever $\langle b, a\rangle \in \varrho$ and $a \neq b$, and transitive if $\langle a, c\rangle \in \varrho$ whenever $\langle a, b\rangle \in \varrho$

and $\langle b, c \rangle \in \varrho$. A binary relation is a preorder if it is reflexive and transitive; a partial ordering if it is reflexive, antisymmetric and transitive; and an equivalence relation if it is reflexive, symmetric and transitive. For an equivalence relation ϱ on A, the sets $a/\varrho = \{\, b \in A : \langle a, b \rangle \in \varrho \,\}$, for all $a \in A$, are called the blocks of ϱ. A partition of A is a set $\{A_0, \ldots, A_{k-1}\}$ of pairwise disjoint subsets of A such that $A = A_0 \cup \cdots \cup A_{k-1}$. Clearly, equivalence relations of A can be identified with partitions of A. For a partial ordering $\leq$ on a set A we use the notation $a < b$ for $a \leq b$ and $a \neq b$, and $a \prec b$ for $a < b$ but for no $c \in A$ does $a < c < b$ hold. Finite partially ordered sets (posets) can be pictured by Hasse diagrams, with the elements depicted as points on a plane, larger elements corresponding to higher points, and the covering relation ($\prec$) represented by ascending straight line segments.

A groupoid $\mathbf{A}$ is idempotent if $aa = a$ for all $a \in A$; commutative if $ab = ba$ for all $a, b \in A$; associative if $a(bc) = (ab)c$ for all $a, b, c \in A$; and conservative if $ab \in \{a, b\}$ for all $a, b \in A$. Commutative conservative groupoids are called tournaments, as described in the introduction. A semilattice is an idempotent, commutative, associative groupoid. Given a semilattice $\mathbf{A}$, one can define a partial ordering $\leq$ on A by letting $a \leq b$ if and only if $ab = a$. A poset $\langle A; \leq \rangle$ is correlated with a semilattice in this fashion if and only if every pair of elements of A has a greatest lower bound with respect to $\leq$.

A lattice is an algebra $\langle A; \wedge, \vee \rangle$ with two binary operations such that both $\langle A; \wedge \rangle$ and $\langle A; \vee \rangle$ are semilattices and for all $x, y \in A$, $x \wedge y = x$ if and only if $x \vee y = y$. Thus, the partial orderings correlated with $\langle A; \wedge \rangle$ and $\langle A; \vee \rangle$ are the inverses of each other. A poset $\langle A; \leq \rangle$ is correlated with a lattice in this fashion if and only if every pair of elements of A has a greatest lower and a least upper bound with respect to $\leq$. If a lattice is finite, then it has a largest and smallest element, usually denoted by 1 and 0, respectively. If a lattice has a smallest element 0, then its covers are called atoms. A lattice $\mathbf{A}$ is distributive if it satisfies the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. We call $\mathbf{A}$ meet semi-distributive, if whenever elements $x, y, z \in A$ satisfy $x \wedge y = x \wedge z$, they also satisfy $x \wedge (y \vee z) = x \wedge y$. An element $a \in A$ is called join irreducible, if for all elements $x, y \in A$, $a = x \vee y$ implies that $a = x$ or $a = y$. Meet irreducible elements are defined analogously.

Let $\mathbf{A} = \langle A; F \rangle$ be an algebra. A binary relation $\varrho \subseteq A^2$ is compatible with $\mathbf{A}$ if $\langle f(a_0, \ldots, a_{n-1}), f(b_0, \ldots, b_{n-1}) \rangle \in \varrho$ for all operations $f \in F$ and for all choices of pairs $\langle a_0, b_0 \rangle, \ldots, \langle a_{n-1}, b_{n-1} \rangle \in \varrho$. Compatible equivalence relations are called congruences. We will typically use lower-case Greek letters to denote congruences. The set of congruences of $\mathbf{A}$ forms a lattice, denoted by $\operatorname{Con} \mathbf{A}$. The smallest and largest congruences of $\mathbf{A}$ are $0_{\mathbf{A}} = \mathrm{id}_{\mathbf{A}}$ and $1_{\mathbf{A}} = A^2$, respectively. The meet $\alpha \wedge \beta$ of two congruences $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ is their intersection $\alpha \cap \beta$, while the join $\alpha \vee \beta$ is the least equivalence relation extending the set $\alpha \cup \beta$. For a pair $\langle a, b \rangle \in A^2$, denote by $\mathrm{Cg}_{\mathbf{A}}(a, b)$ the least congruence of $\mathbf{A}$ containing $\langle a, b \rangle$. An algebra $\mathbf{A}$ is called simple if $\operatorname{Con} \mathbf{A}$ is a two element lattice. This holds if and only if $\mathbf{A}$ has at least two elements and has no congruences other than the trivial ones: $0_{\mathbf{A}}$ and $1_{\mathbf{A}}$. An algebra $\mathbf{A}$ is called subdirectly irreducible if $\operatorname{Con} \mathbf{A}$ has a congruence $\mu \neq 0_{\mathbf{A}}$, called the monolith, such that for every congruence $\alpha \in \operatorname{Con} \mathbf{A}$ either $\alpha = 0_{\mathbf{A}}$ or $\mu \leq \alpha$. Thus $0_{\mathbf{A}}$ is meet irreducible, and μ is a (in fact, the unique) cover of $0_{\mathbf{A}}$ in the lattice $\operatorname{Con} \mathbf{A}$.

Two algebras $\mathbf{A}$ and $\mathbf{B}$ are called similar if their sets of operations are indexed over the same set and their corresponding operations are of equal arity. A mapping $\varphi : A \to B$ between two similar algebras $\langle A; F \rangle$ and $\langle B; F \rangle$ is a homomorphism if $\varphi(f^{\mathbf{A}}(a_0, \ldots, a_{n-1})) = f^{\mathbf{B}}(\varphi(a_0), \ldots, \varphi(a_{n-1}))$ for all operations $f \in F$ and for all elements $a_0, \ldots, a_{n-1} \in A$. The kernel of a homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ is the congruence of $\mathbf{A}$ defined as $\ker \varphi = \{ \langle a, b \rangle \in A^2 : \varphi(a) = \varphi(b) \}$. A homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ is called an isomorphism if it is one-to-one and onto, an endomorphism if $A = B$, and an automorphism if it is both an isomorphism and an endomorphism. To denote that $\mathbf{A}$ isomorphic to $\mathbf{B}$, that is, there exists an isomorphism between them, we write $\mathbf{A} \cong \mathbf{B}$. We say that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$ if there exists a homomorphism of $\mathbf{A}$ onto $\mathbf{B}$.

Given an algebra $\mathbf{A} = \langle A; F\rangle$ and a congruence $\alpha \in \operatorname{Con}\mathbf{A}$, the factor algebra $\mathbf{A}/\alpha = \langle A/\alpha; F\rangle$ is defined on the set $A/\alpha = \{\, a/\alpha : a \in A \,\}$ of blocks of α by setting $f^{\mathbf{A}/\alpha}(a_0/\alpha, \ldots, a_{n-1}/\alpha) = f^{\mathbf{A}}(a_0, \ldots, a_{n-1})/\alpha$ for all operations $f \in F$ and elements $a_0/\alpha, \ldots, a_{n-1}/\alpha \in A/\alpha$. It is not hard to see that the operations $f^{\mathbf{A}/\alpha}$ are well-defined, because α is a compatible equivalence relation. The mapping $\varphi : \mathbf{A} \to \mathbf{A}/\alpha$ defined by $\varphi(a) = a/\alpha$ is a homomorphism, called the natural homomorphism.

Let $\mathbf{A} = \langle A; F\rangle$ and $\mathbf{B} = \langle B; F\rangle$ be two similar algebras. We call $\mathbf{B}$ a subalgebra of $\mathbf{A}$, and write $\mathbf{B} \leq \mathbf{A}$, if $B \subseteq A$ and $f^{\mathbf{B}}(a_0, \ldots, a_{n-1}) = f^{\mathbf{A}}(a_0, \ldots, a_{n-1})$ for all operations $f \in F$ and for all elements $a_0, \ldots, a_{n-1} \in B$. A subset $C \subseteq A$ is called a subuniverse of $\mathbf{A}$ if it is closed under all operations of $\mathbf{A}$, that is, $f(a_0, \ldots, a_{n-1}) \in B$ for all $f \in F$ and for all elements $a_0, \ldots, a_{n-1} \in C$. Clearly, there is a one-to-one correspondence between subalgebras and nonvoid subuniverses of $\mathbf{A}$. The subalgebra of $\mathbf{A}$ generated by a set $C_0 \subseteq A$ is the smallest subalgebra $\mathbf{C} \leq \mathbf{A}$ such that $C_0 \subseteq C$. An algebra $\mathbf{A}$ is generated by a set $A_0 \subseteq A$ if the subalgebra generated by A_0 is $\mathbf{A}$. We call an algebra n-generated, or finitely generated, if it is generated by an n-element set, or a finite set, respectively.

Let $\{\, \mathbf{A}_i : i \in I \,\}$ be a set of similar algebras, and let B be the Cartesian product $\prod_{i\in I} A_i$ of the sets A_i. The i-th projection, $i \in I$, is the mapping $\pi_i : B \to A_i$ defined by $\pi_i(\bar{a}) = a_i$ for all tuples $\bar{a} = \langle \ldots, a_i, \ldots\rangle \in \prod_{i\in I} A_i$. For a subset $K \subseteq I$, define the mapping $\pi_K : B \to \prod_{k\in K} A_k$ by $\pi_K(\bar{a}) = \langle a_k : k \in K\rangle$. The product $\mathbf{B} = \prod_{i\in I} \mathbf{A}_i$ is the algebra defined on B by setting $\pi_j(f^{\mathbf{B}}(\bar{a}_0, \ldots, \bar{a}_{n-1})) = f^{\mathbf{A}_j}(\pi_j(\bar{a}_0), \ldots, \pi_j(\bar{a}_{n-1}))$ for all operations $f \in F$, for all tuples $\bar{a}_0, \ldots, \bar{a}_{n-1} \in \prod_{i\in I} A_i$ and for all indices $j \in I$. Clearly, all projections of $\mathbf{B}$ are onto homomorphisms, by definition. We say that an algebra $\mathbf{C}$ is a subdirect product of the algebras $\mathbf{A}_i$ $(i \in I)$, if $\mathbf{C} \leq \mathbf{B}$ and every projection π_i maps C onto A_i. It is not hard to see that an algebra is subdirectly irreducible if it is not isomorphic in a non-trivial way to a subdirect product. Moreover, the subdirect representation theorem of Birkhoff states that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.

By a language $\mathcal{L}$ we shall mean an indexed set $F = \{ f_i : i \in I \}$ of operation symbols such that a nonnegative integer, called the arity, is assigned to each member of F. An algebra $\mathbf{A}$ is in the language $\mathcal{L}$ if the set of operations of $\mathbf{A}$ is indexed over I and the corresponding operations and operation symbols are of equal arity. Clearly, algebras are similar if and only if they are in the same language. For any nonvoid set X, there is an algebra $\mathbf{F}_{\mathcal{L}}(X)$ in $\mathcal{L}$, generated by X, having the property that every mapping φ_0 of X into an algebra $\mathbf{A}$ in $\mathcal{L}$ has a unique extension φ which is a homomorphism of $\mathbf{F}_{\mathcal{L}}(X)$ into $\mathbf{A}$. The algebra $\mathbf{F}_{\mathcal{L}}(X)$ is called the free algebra in $\mathcal{L}$, freely generated by X. It is determined up to isomorphism by X.

A term in a language $\mathcal{L}$ is simply a member of $\mathbf{F}_{\mathcal{L}}(X)$ for some finite set X. Terms $t \in \mathbf{F}_{\mathcal{L}}(X)$ where $X = \{x_0, \ldots, x_{n-1}\}$ will be written as $t(x_0, \ldots, x_{n-1})$. Let $t = t(x_0, \ldots, x_{n-1})$ be such a term. Given elements $a_0, \ldots, a_{n-1}$ in an algebra $\mathbf{A}$ in $\mathcal{L}$, we define $t^{\mathbf{A}}(a_0, \ldots, a_{n-1})$ to be the element $\varphi(t)$ where φ is the homomorphism of $\mathbf{F}_{\mathcal{L}}(X)$ into $\mathbf{A}$ with $\varphi(x_0) = a_0, \ldots, \varphi(x_{k-1}) = a_{k-1}$. This defines a n-ary operation $t^{\mathbf{A}}$ on the universe of $\mathbf{A}$, corresponding to the term $t(x_0, \ldots, x_{n-1})$. Operations in the algebra $\mathbf{A}$ that can be defined in this way are called term operations of $\mathbf{A}$. Given a term operation $t^{\mathbf{A}}(x_0, \ldots, x_{n-1})$ of $\mathbf{A}$, a non-negative integer $k \leq n$, and elements $c_k, \ldots, c_{n-1} \in A$, we define an operation $p^{\mathbf{A}}(x_0, \ldots, x_{k-1}) = t^{\mathbf{A}}(x_0, \ldots, x_{k-1}, c_k, \ldots, c_{n-1})$ of $\mathbf{A}$. Operations that can be defined in this way are called polynomials of $\mathbf{A}$.

An equation in the language $\mathcal{L}$ is an ordered pair of terms, both of which are members of the same free algebra. Equations are written in the form $s(x_0, \ldots, x_{n-1}) = t(x_0, \ldots, x_{n-1})$. Such an equation is said to be an identity of an algebra $\mathbf{A}$ in $\mathcal{L}$ if $s^{\mathbf{A}} = t^{\mathbf{A}}$ (an equivalent expression is that $s = t$ holds in $\mathbf{A}$). If Σ is a set of equations in the language $\mathcal{L}$, the class of all algebras in $\mathcal{L}$ in which every member of Σ is an identity will be called the class of models of Σ. Classes of algebras of this form are called varieties. It is well known that the class of all groups, all rings, all semilattices, and all lattices each forms a variety. We say that a variety $\mathcal{V}$ is finitely based if it is the class of models of some finite set of equations.

For any class $\mathcal{K}$ of similar algebras, $\mathcal{H}(\mathcal{K})$, $\mathcal{S}(\mathcal{K})$, and $\mathcal{P}(\mathcal{K})$ denote the class of all algebras that are, respectively, homomorphic images of algebras in $\mathcal{K}$, isomorphic to a subalgebra of an algebra in $\mathcal{K}$, or isomorphic to a product of algebras in $\mathcal{K}$. According to the HSP-theorem of Birkhoff, a class $\mathcal{K}$ of similar algebras is a variety if $\mathcal{K} = \mathcal{HSP}(\mathcal{K})$, and the smallest variety containing a class $\mathcal{K}$ of similar algebras is $\mathcal{V}(\mathcal{K}) = \mathcal{HSP}(\mathcal{K})$. We call a variety $\mathcal{V}$ finitely generated if $\mathcal{V} = \mathcal{V}(\mathbf{A})$ for some finite algebra $\mathbf{A}$. A variety $\mathcal{V}$ is congruence distributive (congruence meet-semidistributive) if the congruence lattice of every member of $\mathcal{V}$ is distributive (meet-semidistributive). A variety $\mathcal{V}$ is said to be locally finite if every finitely generated subalgebra of a member of $\mathcal{V}$ is finite. A variety $\mathcal{W}$ is a subvariety of $\mathcal{V}$ if $\mathcal{W} \subseteq \mathcal{V}$. Subvarieties of $\mathcal{V}$ form a lattice, called the subvariety lattice of $\mathcal{V}$.

Let $\mathcal{L}$ be a language, and $\mathcal{V}$ be a variety of algebras in $\mathcal{L}$. For any nonvoid set X there exists an algebra $\mathbf{F}_{\mathcal{V}}(X)$ in $\mathcal{V}$, generated by X, such that every mapping of X into an algebra $\mathbf{A} \in \mathcal{V}$ uniquely extends to a homomorphism of $\mathbf{F}_{\mathcal{V}}(X)$ into $\mathbf{A}$. We call $\mathbf{F}_{\mathcal{V}}(X)$ the free algebra in $\mathcal{V}$, freely generated by X. It is determined (in $\mathcal{V}$) up to isomorphism by X. Clearly, $\mathbf{F}_{\mathcal{V}}(X) \cong \mathbf{F}_{\mathcal{L}}(X)/\vartheta$ for some congruence ϑ of $\mathbf{F}_{\mathcal{L}}(X)$. One proof of Birkhoff's theorem proceeds by noting that, for all pairs $s, t \in \mathbf{F}_{\mathcal{L}}(X)$ of terms, the equation $s = t$ holds in all members of $\mathcal{V}$ if and only if $\langle s, t\rangle \in \vartheta$.

CHAPTER III

PRELIMINARY RESULTS

$\mathcal{T}$ is locally finite

Theorem 3.1. *Let* $\mathbf{A} \in \mathcal{T}$ *be an* n*-generated algebra that is a homomorphic image of a subalgebra* $\mathbf{B}$ *of a product of tournaments. Then there exists a finite subalgebra* $\mathbf{C} \leq \mathbf{B}$ *such that* $\mathbf{A}$ *is a homomorphic image of* $\mathbf{C}$*, and* $\mathbf{C}$ *is a subdirect product of finitely many at most* n*-element tournaments.*

Proof. Let $\mathbf{B}$ be a subalgebra of a product $\prod_{i<\kappa} \mathbf{T}_i$ of tournaments $\mathbf{T}_i$, and φ be a homomorphism of $\mathbf{B}$ onto $\mathbf{A}$. Let A_0 be an n-element generating set for $\mathbf{A}$, and for each $a \in A_0$ take a representative element $b \in B$ such that $\varphi(b) = a$, and let C_0 be the set of these representative elements. Clearly, $|C_0| = n$. Denote by $\mathbf{C}$ the subalgebra of $\mathbf{B}$ generated by C_0. Since A_0 is a generating set for $\mathbf{A}$ and $\varphi(C_0) = A_0$, the homomorphism $\varphi|_{\mathbf{C}}$ maps $\mathbf{C}$ onto $\mathbf{A}$. We argue that $\mathbf{C}$ is a subdirect product of finitely many at most n-element tournaments. Consider the projection π_i of $\mathbf{C}$ into the tournament $\mathbf{T}_i$. The subtournament $\mathbf{S}_i = \pi_i(\mathbf{C})$ of $\mathbf{T}_i$ is generated by $\pi_i(C_0)$. Since every nontrivial subset of a tournament is a subuniverse, $S_i = \pi_i(C_0)$ and $|S_i| \leq n$. Accordingly, $\mathbf{C} \leq \prod_{i<\kappa} \mathbf{S}_i$ is a subdirect product of at most n-element tournaments. But there are only finitely many at most n-element tournaments, up to isomorphism. And for each such tournament $\mathbf{S}$ there are only finitely many homomorphisms π of $\mathbf{C}$ onto $\mathbf{S}$, because π is uniquely determined by its value on C_0. Therefore, $\mathbf{C}$ is a subdirect product of a finite subset of $\{\, \mathbf{S}_i : i < \kappa \,\}$. $\square$

Corollary 3.2. *The variety* $\mathcal{T}$ *is locally finite. The* n*-generated free algebra in* $\mathcal{T}$ *is isomorphic to a subdirect product of at most* n*-element tournaments.* $\square$

Three variable equations of $\mathcal{T}$

For a positive integer, we denote by $\mathcal{T}^n$ the variety of groupoids determined by all of the equations in at most n variables that are satisfied in $\mathcal{T}$. In this way we obtain a chain $\mathcal{T}^1 \supseteq \mathcal{T}^2 \supseteq \mathcal{T}^3 \supseteq \cdots \supseteq \mathcal{T}$ of varieties such that $\bigcap_{i=1}^{\infty} \mathcal{T}^i = \mathcal{T}$. It is not hard to see that $\mathcal{T}^2$ is just the variety of commutative idempotent groupoids.

Theorem 3.3. *The following four equations are a base for the equational theory of* $\mathcal{T}^3$*:*

(1) $xx = x$

(2) $xy = yx$

(3) $xy \cdot x = xy$

(4) $(xy \cdot xz)(xy \cdot yz) = xyz$

In particular, the following equations are consequences of $(1) - (4)$*:*

(5) $(xy \cdot xz)x = xy \cdot xz$

(6) $xyz \cdot xz = yxzx$

(7) $(xy \cdot xz)(xyz) = xyz$

(8) $(xy \cdot xz)z = xyz$

(9) $(xy \cdot xz) \cdot yz = xyzy$

(10) $xyzy = xzyz$

(11) $xyzy \cdot xy = yx \cdot yz$

(12) $(xy \cdot xz)(yzx) = xy \cdot xz$

(13) $xyzy \cdot yz = xyzy$

(14) $xyzy \cdot (xy \cdot xz) = xyzy$

(15) $xyz \cdot xzy = xyzy$

(16) $xyzy \cdot (yx \cdot yz) = yx \cdot yz$

(17) $xyzy \cdot yxzx = zx \cdot zy$

(18) $xyzyx = yzx$

(19) $xyzy \cdot yzx = yzx$

The free, 3*-generated groupoid* $\mathbf{F}_3$ *has 15 elements*

$a = x$	$d = xy$	$g = yzx$	$j = xy \cdot xz$	$m = yxzx = yzxz$
$b = y$	$e = xz$	$h = xzy$	$k = yx \cdot yz$	$n = zxyx = zyxy$
$c = z$	$f = yz$	$i = xyz$	$l = zx \cdot zy$	$o = xyzy = xzyz$

The multiplication table for $\mathbf{F}_3$ *is given below.*

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
a	a	d	e	d	e	g	g	n	m	j	g	g	m	n	g
b	d	b	f	d	h	f	n	h	o	h	k	h	h	n	o
c	e	f	c	i	e	f	m	o	i	i	i	l	m	i	o
d	d	d	i	d	j	k	n	n	i	j	k	n	j	n	k
e	e	h	e	j	e	l	m	h	m	j	m	l	m	j	l
f	g	f	f	k	l	f	g	o	o	o	k	l	l	k	o
g	g	n	m	n	m	g	g	n	m	j	g	g	m	n	g
h	n	h	o	n	h	o	n	h	o	h	k	h	h	n	o
i	m	o	i	i	m	o	m	o	i	i	i	l	m	i	o
j	j	h	i	j	j	o	j	h	i	j	i	h	j	j	o
k	g	k	i	k	m	k	g	k	i	i	k	g	m	k	k
l	g	h	l	n	l	l	g	h	l	h	g	l	l	n	l
m	m	h	m	j	m	l	m	h	m	j	m	l	m	j	l
n	n	n	i	n	j	k	n	n	i	j	k	n	j	n	k
o	g	o	o	k	l	o	g	o	o	o	k	l	l	k	o

Proof. It is easy to check that equations (1) − (4) are valid in all tournaments. We prove the other equations from these. Put $u = xy$, $v = xz$ and $w = yz$.

(5) $(xy \cdot xz)x =_{(4)} ((xy \cdot xz)(xy \cdot x)) \cdot ((xy \cdot xz)(xz \cdot x)) =_{(1,2,3)} xy \cdot xz$

(6) $yxzx =_{(2)} xyzx =_{(4)} (xyz \cdot xyx)(xyz \cdot zx) =_{(2,3)} xyz \cdot xz$

(7) $(xy \cdot xz) \cdot xyz =_{(4)} (xy \cdot xz) \cdot ((xy \cdot xz)(xy \cdot yz)) =_{(2,3)} (xy \cdot xz)(xy \cdot yz) =_{(4)} xyz$

(8) $(xy \cdot xz)z =_{(4)} ((xy \cdot xz) \cdot xyz) \cdot ((xy \cdot xz) \cdot xzz) =_{(2,3)} (xy \cdot xz) \cdot xyz =_{(7)} xyz$

(9) $xyzy =_{(6)} yxz \cdot yz =_{(4)} (yx \cdot yz)(yx \cdot xz) \cdot yz =_{(2)} (uv \cdot uw) \cdot w =_{(8)} uvw = (xy \cdot xz) \cdot yz$

(10) $xyzy =_{(9)} (xy \cdot xz) \cdot yz =_{(2)} (xz \cdot xy) \cdot zy =_{(9)} xzyz$

(11) $xyzy \cdot xy =_{(2)} zuyu =_{(10)} zyuy =_{(2)} (yx \cdot yz)y =_{(5)} yx \cdot yz$

(12) $(xy \cdot xz) \cdot yzx =_{(4)} (xy \cdot xz) \cdot ((yz \cdot yx)(yz \cdot zx)) =_{(2)} (wu \cdot wv) \cdot uv =_{(9)} wuvu =_{(2)} ((yz \cdot yx) \cdot zx) \cdot yx =_{(9)} yzxz \cdot yx =_{(10)} yxzx \cdot yx =_{(11)} xy \cdot xz$

(13) $xyzy \cdot yz =_{(9)} ((xy \cdot xz) \cdot yz) \cdot yz =_{(3)} (xy \cdot xz) \cdot yz =_{(9)} xyzy$

(14) $xyzy \cdot (xy \cdot xz) =_{(9)} ((xy \cdot xz) \cdot yz) \cdot (xy \cdot xz) =_{(2,3)} (xy \cdot xz) \cdot yz =_{(9)} xyzy$

(15) $xyz \cdot xzy =_{(4)} ((xy \cdot xz)(xy \cdot yz)) \cdot ((xz \cdot xy)(xz \cdot zy)) =_{(2)} (uv \cdot uw)(uv \cdot vw) =_{(4)} uvw = (xy \cdot xz) \cdot yz =_{(9)} xyzy$

(16) $xyzy \cdot (yx \cdot yz) =_{(2)} (zy \cdot xy) \cdot (z \cdot xy \cdot y) = zyu \cdot zuy =_{(15)} zyuy =_{(2)} (yx \cdot yz) \cdot y =_{(5)} yx \cdot yz$

(17) $xyzy \cdot yxzx =_{(9)} ((xy \cdot xz) \cdot yz) \cdot ((yx \cdot yz) \cdot xz) =_{(2)} uwv \cdot uvw =_{(15)} uwvw =_{(6)} wuv \cdot wv =_{(2)} ((zx \cdot zy) \cdot xy) \cdot (zx \cdot zy) =_{(9)} xzyz \cdot (zx \cdot zy) =_{(16)} zx \cdot zy$

(18) $xyzyx =_{(8)} (xyzy \cdot xyzx) \cdot x =_{(2)} (xyzy \cdot yxzx) \cdot x =_{(17)} (zx \cdot zy) \cdot x =_{(2)} (zy \cdot zx) \cdot x =_{(8)} zyx =_{(2)} yzx$

(19) $xyzy \cdot yzx =_{(9,4)} ((xy \cdot xz) \cdot yz) \cdot ((yz \cdot yx)(yz \cdot zx)) =_{(2,1)} (wu \cdot wv) \cdot uvw =_{(12)}$
$wu \cdot wv =_{(2)} (yz \cdot yx)(yz \cdot zx) =_{(4)} yzx$

Now that the equations are proved, we can start to build the free groupoid on three generators x, y, z. Equations (1) – (19) imply that the fifteen terms $a, \ldots, o$ multiply among each other, with respect to the equational theory of $\mathcal{T}^3$, as in the table. Consequently, the free groupoid can have no more than fifteen elements. Clearly, $a, \ldots, f$ are distinct from each other and from each of the elements $g, \ldots, o$. The last nine elements are also distinct from each other: one can easily check that the terms behave differently on the three-element cycle. □

It is not easy to picture the free groupoid $\mathbf{F}_3$. For one possible graphical representation of $\mathbf{F}_3$ see Fig. 2 in which not all pairs of comparable elements are connected. However, the missing ones can be recovered with some practice.

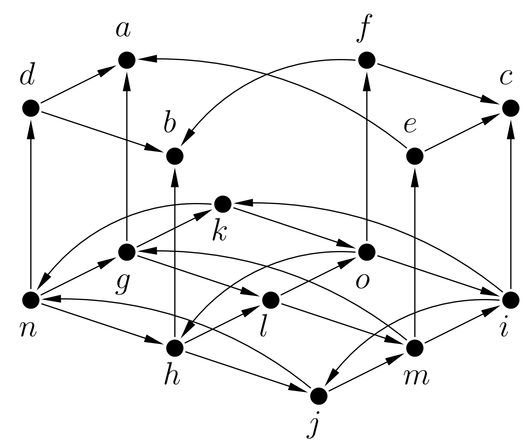

Figure 2: The free groupoid $\mathbf{F}_3$

Lemma 3.4. *Let $\mathbf{A} \in \mathcal{T}^3$ and let $a, b, c \in A$. Then:*

(1) *If $a \to c$ and $b \to c$, then $ab \to c$.*

(2) *If $c \to a$ and $c \to b$, then $c \to ab$.*

(3) *If* $ab \to c \to a$, *then* $bc = ab$.

(4) *If* $a \to c$ *and* $c \to b$, *then* $a \to abc \to b$.

(5) *If* $a \to b \to c \to a$, *then either* $a = b = c$ *or* $a \neq b \neq c \neq a$.

(6) *If* $a \to c \to b$, $a \not\to b$, *and* ab *and* c *are comparable, then* $c \to ab$.

Proof. Each of the situations (1) – (4) generates a congruence ϑ of the free algebra $\mathbf{F}_3$ with generators a, b, c. The congruence ϑ can be easily described from the multiplication table of $\mathbf{F}_3$, and then the conclusion can be verified in $\mathbf{F}_3/\vartheta$. For (4) we get the groupoid pictured in Fig. 3. Statement (5) follows from the fact that the three element cycle is simple. To check (6), consider the groupoid pictured in Fig. 3. Since $a \not\to b$, that is, $ab \neq a$, we get $a \neq ab \neq abc \neq a$ by (5). But ab and c are comparable, hence we must have $abc = c$. Then $c = abc \to ab$. $\square$

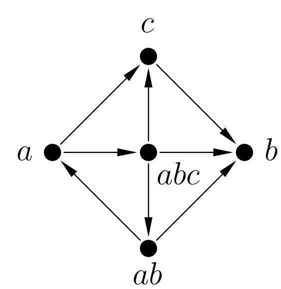

Figure 3: The groupoid $\mathbf{F}_3/\vartheta$ where $\vartheta = \mathrm{Cg}(a, ac) \vee \mathrm{Cg}(c, cb)$

$\mathcal{T}$ is non-finitely based

Theorem 3.5. *For every* $n \geq 3$ *there exists a groupoid* $\mathbf{M}_n$ *with* $n+2$ *elements such that* $\mathbf{M}_n$ *belongs to* $\mathcal{T}^n$ *but not to* $\mathcal{T}^{n+1}$. *Consequently,* $\mathcal{T}$ *is not finitely based.*

Proof. Put $M_n = \{a, b_0, \ldots, b_n\}$ and define a commutative and idempotent multiplication on M_n by

$$ab_1 = b_0,$$

$$ab_i = b_i \text{ for } i \leq n-1 \text{ and } i \neq 1,$$

$$ab_n = a,$$

$$b_i b_{i+1} = b_i \text{ for } i < n-1,$$

$$b_n b_{n-1} = b_n,$$

$$b_i b_j = b_{\max(i,j)} \text{ for } |i-j| \geq 2 \text{ and } i, j < n,$$

$$b_n b_i = b_i \text{ for } i < n-1;$$

the multiplication in the other cases is given by commutativity and idempotency (see Fig. 4).

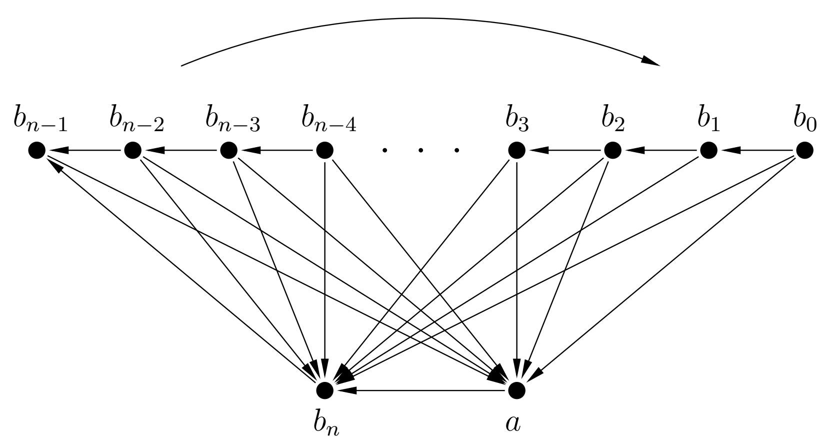

Figure 4: The groupoid $\mathbf{M}_n$

Define terms $t_1, s_1, t_2, s_2, \ldots, t_n, s_n$ in $n+1$ variables $x, y_1, \ldots, y_n$ as follows:

$t_1 = y_1$ and $s_1 = xy_1$;

$t_i = s_{i-1}y_i$ and $s_i = t_{i-1}y_i$ for $2 \leq i \leq n-1$;

$t_n = t_{n-1}y_{n-3}y_nt_{n-1}$ and $s_n = s_{n-1}y_{n-3}y_nt_{n-1}$ if $n \geq 4$,

while $t_3 = t_2s_1y_3t_2$ and $s_3 = s_2s_1y_3t_2$ if $n = 3$.

Finally, put $t = s_1t_ns_nt_n(xt_n)$ and $s = t(s_1t_n)$.

We are going to prove that the equation $t = s$ is satisfied in any tournament. There will be no confusion if we do not distinguish between a term and its value in a tournament under an interpretation. We consider two cases:

If $s_1 = x$, then

$$t = xt_ns_nt_n(xt_n)$$

and

$$s = xt_ns_nt_n(xt_n)(xt_n) = xt_ns_nt_n(xt_n) = t.$$

The other case is $s_1 = y_1$. Then we have $t_1 = s_1, t_2 = s_2, \ldots, t_n = s_n$. Consequently,

$$t = y_1t_n(xt_n) \quad \text{and} \quad s = y_1t_n(xt_n)(y_1t_n); \tag{$*$}$$

clearly, these two values are equal. (In these arguments we have repeatedly used equation (3) from Theorem 3.3.)

So, $t = s$ in every tournament under any interpretation.

This means that the equation $t = s$ is satisfied in $\mathcal{T}$. On the other hand, we are going to show that the equation is not satisfied in the groupoid $\mathbf{M}_n$. Consider the interpretation $x \mapsto a$, $y_i \mapsto b_i$. By induction on $i = 1, \ldots, n$ we can see that $t_i \mapsto b_i$ and $s_i \mapsto b_{i-1}$. So, $t \mapsto a$ and $s \mapsto b_0$. Since $a \neq b_0$, the equation $t = s$ is not satisfied in $\mathbf{M}_n$.

We have proved that the groupoid $\mathbf{M}_n$ does not belong to $\mathcal{T}$. Since it is generated by $n+1$ elements, it follows that it does not belong to $\mathcal{T}^{n+1}$. In order to prove that it belongs to $\mathcal{T}^n$, it is sufficient to show that every subgroupoid of $\mathbf{M}_n$ generated by at most n elements belongs to $\mathcal{T}$.

If we remove either a or b_1 from $\mathbf{M}_n$, we obtain a subtournament. If we remove b_0, we must remove either a or b_1 in order to obtain a subgroupoid. So, it is sufficient to prove that, for any $i = 2, \dots, n$, $\mathbf{M}_n \setminus \{b_i\}$ is a subgroupoid belonging to $\mathcal{T}$. One can easily check that there are two congruences C_1 and C_2 of $\mathbf{M}_n \setminus \{b_i\}$ with trivial intersection, such that both factors $(\mathbf{M}_n \setminus \{b_i\})/C_1$ and $(\mathbf{M}_n \setminus \{b_i\})/C_2$ are tournaments: C_1 is the congruence generated by (a, b_0) and C_2 is the congruence generated by (b_1, b_0). (It is easy to see that $\{a, b_0\}$ and $\{b_{i-1}, \dots, b_0\}$ are the only non-singleton blocks of C_1 and C_2, respectively.) Consequently, $\mathbf{M}_n \setminus \{b_i\}$ is a subdirect product of two tournaments (its factor groupoids by C_1 and C_2) and hence belongs to $\mathcal{T}$. $\square$

Now we show that $\mathcal{T}$ is inherently non-finitely generated. We use often the tournament $\mathbf{L}_n$, which consists of n elements $a_0, \dots, a_{n-1}$ with $a_i \to a_j$ if and only if either $i = j$ or $j = i+1$ or $i > j+1$. Let $\mathbf{N}_n$ be the tournament $\mathbf{L}_n$ with two elements a and b adjoined where $a_i \to a \to b$ for all $i < n$, $a_i \to b$ for all $i < n-1$ and $b \to a_{n-1}$ (see Fig. 5).

$\mathcal{T}$ is inherently non-finitely generated

Theorem 3.6. *If $\mathbf{A}$ is any groupoid with $\mathbf{N}_n \in \mathcal{V}(\mathbf{A})$ then $|A| \geq n$. Hence the variety $\mathcal{T}$ is inherently non-finitely generated.*

Proof. We can assume that $\mathbf{A}$ is finite, $\mathbf{D}$ is a subalgebra of A^k, φ is a homomorphism of $\mathbf{D}$ onto $\mathbf{N}_n$, and k is minimum for the existence of $\mathbf{D}$ and φ. Thus there exist

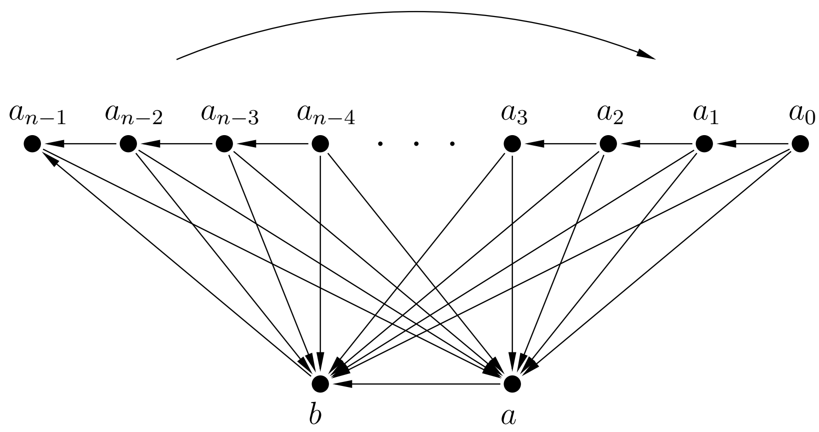

Figure 5: The tournament $\mathbf{N}_n$

$f, g \in D$ such that $\varphi(f) \neq \varphi(g)$ and $f|_{k-1} = g|_{k-1}$.

The crucial property of $\mathbf{L}_n$ is that for any $x \neq y$ and $u \neq v$ in $\mathbf{L}_n$ there is a translation (i.e., a polynomial p of the form $p(w) = wr_1r_2 \cdots r_t$) such that $\{p(x), p(y)\} = \{u, v\}$. In fact, $\mathbf{L}_n$ is a simple algebra of type $\mathbf{3}$ and it follows from a result in [10] that $\mathbf{L}_n$ must be a homomorphic image of a subalgebra of $\mathbf{A}$ (actually, $k = 1$). But let's just prove directly that $|A| \geq n$.

From the two remarks above, there must exist $f_i, g_i \in D$ such that $f_i|_{k-1} = g_i|_{k-1}$ and $\varphi(f_i) = a$ and $\varphi(g_i) = a_i$. Then put

$$f = f_0 f_1 \ldots f_{n-1},$$

$$h_i = f_0 \ldots f_{i-1} g_i f_{i+1} \ldots f_{n-1}$$

and we have that all elements $f, h_0, \ldots, h_{n-1}$ agree on $k - 1$ and $\varphi(f) = a$ while $\varphi(h_i) = a_i$.

These elements of $\mathbf{D}$ must all disagree at their last coordinate, hence $\mathbf{A}$ has at least $n + 1$ elements. $\square$

$\mathcal{T}$ is congruence meet-semidistributive

Theorem 3.7. *The variety $\mathcal{T}$ is congruence meet-semidistributive.*

Proof. We will prove the theorem using basic tame congruence theory. Alternatively, one could use the Mal'cev condition given in [21].

By Exercise 7.14 (4) of [10], a locally finite variety $\mathcal{V}$ is congruence meet-semidistributive if and only if $\mathrm{typ}\{\mathcal{V}\} \cap \{\mathbf{1}, \mathbf{2}\} = \emptyset$. Recall that $\mathcal{T}$ is locally finite. To check that types $\mathbf{1}$ and $\mathbf{2}$ are omitted in $\mathcal{T}$, take a finite algebra $\mathbf{A} \in \mathcal{T}$, a prime congruence quotient $\alpha \prec \beta$ of $\mathbf{A}$, and an $\langle \alpha, \beta \rangle$-subtrace $\{a, b\}$. Thus $\langle a, b \rangle \in \beta \setminus \alpha$. Clearly, both pairs $\langle a, ab \rangle$ and $\langle ab, b \rangle$ of elements of A are β related, and at least one of them is not α related. Assume, for example, that $\langle a, ab \rangle \in \beta \setminus \alpha$. Then $\{a, ab\}$ is an $\langle \alpha, \beta \rangle$-subtrace, and multiplication is a semilattice operation on $\{a, ab\}$. Therefore, $\mathrm{typ}(\alpha, \beta) \notin \{\mathbf{1}, \mathbf{2}\}$, by Exercise 5.11 (1) of [10]. □

Infinitely many incomparable tournaments

In the rest of this chapter we construct an infinite sequence of finite simple tournaments $\mathbf{A}_n$ ($n \geq 8$) such that no one is isomorphic to a subalgebra of some other one. The tournament $\mathbf{A}_n$ is defined on the set $A_n = \{a_{n,1}, \dots, a_{n,n}\}$ in the following way (see Fig. 6):

$$a_{n,n} \to a_{n,1};$$

$$a_{n,i+2} \to a_{n,i} \text{ for } 1 \leq i \leq n-2;$$

$$a_{n,i} \to a_{n,j} \text{ for } 1 \leq i < j \leq n,\ j \neq i+2,\ (i,j) \neq (1,n).$$

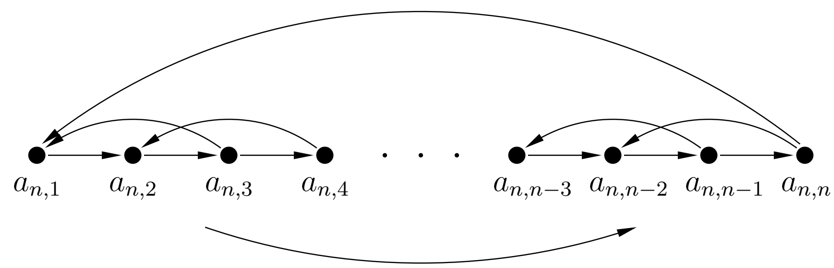

Figure 6: The tournament $\mathbf{A}_n$

Lemma 3.8. *Let $a_{n,i}, a_{n,j}$ be two distinct elements of A_n such that $a_{n,i} \to a_{n,j}$. Put $X = \{x \in A_n \setminus \{a_{n,i}, a_{n,j}\} : a_{n,j} \to x \to a_{n,i}\}$. Then:*

(1) *For $(i, j) = (n, 1)$, $X = \{a_{n,2}, a_{n,4}, a_{n,5}, \ldots, a_{n,n-4}, a_{n,n-3}, a_{n,n-1}\}$ and $|X| \geq 4$.*

(2) *For $1 \leq j < i = j + 2 \leq n$, $X \subseteq \{a_{n,j-2}, a_{n,j+1}, a_{n,j+4}\}$.*

(3) *For $1 = i < j$, $X \subseteq \{a_{n,3}, a_{n,n}\}$.*

(4) *For $i < j = n$, $X \subseteq \{a_{n,1}, a_{n,n-2}\}$.*

(5) *For $2 \leq i < i + 1 = j \leq n - 1$, $X \subseteq \{a_{n,i-1}, a_{n,i+2}\}$.*

(6) *For $2 \leq i < i + 4 = j \leq n - 1$, $X = \{a_{n,i+2}\}$.*

(7) *In all other cases, $X = \emptyset$.*

Proof. It is easy. □

Lemma 3.9. *Let $n, m \geq 8$ and let α be an embedding of $\mathbf{A}_n$ into $\mathbf{A}_m$. Then $\alpha(a_{n,1}) = a_{m,1}$ and $\alpha(a_{n,n}) = a_{m,m}$.*

Proof. We have $\alpha(a_{n,n}) \to \alpha(a_{n,1})$ and, by Lemma 3.8 (1), there are at least four elements $x \in A_m \setminus \{\alpha(a_{n,1}), \alpha(a_{n,n})\}$ such that $\alpha(a_{n,1}) \to x \to \alpha(a_{n,n})$. By Lemma 3.8, it follows that $(\alpha(a_{n,n}), \alpha(a_{n,1})) = (a_{m,m}, a_{m,1})$. □

Lemma 3.10. *Let $n, m \geq 8$ and let α be an embedding of $\mathbf{A}_n$ into $\mathbf{A}_m$. Then $\alpha(a_{n,2}) = a_{m,2}$ and $\alpha(a_{n,3}) = a_{m,3}$.*

Proof. Put $x = \alpha(a_{n,2})$, $y = \alpha(a_{n,3})$, $z = \alpha(a_{n,4})$ and $u = \alpha(a_{n,5})$. Then x, y, z, u are four distinct elements of $A_m \setminus \{a_{m,1}, a_{m,m}\}$ such that $a_{m,1} \to x \to y \to z \to u$, $y \to a_{m,1}$, $z \to x$, $u \to y$, $x \to u$, $a_{m,1} \to u$. From $a_{m,1} \to x \to y \to a_{m,1}$ we get either $(x, y) = (a_{m,2}, a_{m,3})$ or $(x, y) = (a_{m,5}, a_{m,3})$. In the first case we are done, so suppose that $x = a_{m,5}$ and $y = a_{m,3}$. From $y \to z \to x$ (i.e., $a_{m,3} \to z \to a_{m,5}$) we get either $z = a_{m,4}$ or $z = a_{m,7}$.

Suppose $z = a_{m,4}$. From $z \to u \to y$ we get either $u = a_{m,2}$ or $u = a_{m,5}$. In the first case we get a contradiction with $x \to u$, and the second case contradicts $x \neq u$. So, it remains to consider the case $z = a_{m,7}$. From $z \to u \to y$ we get $u = a_{m,5}$, a contradiction with $x \neq u$. $\square$

Lemma 3.11. *Let $n, m \geq 8$ and let α be an embedding of $\mathbf{A}_n$ into $\mathbf{A}_m$. Then $\alpha(a_{n,i}) = a_{m,i}$ for all $i = 1, \ldots, n$.*

Proof. By Lemma 3.9 and Lemma 3.10, this is true for $i = 1, 2, 3$. Let $i \geq 4$ and suppose $\alpha(a_{n,j}) = a_{m,j}$ for all $j < i$. Put $x = \alpha(a_{n,i})$. We have $a_{n,i-1} \to a_{n,i} \to a_{n,i-2}$ in A_n, and thus $a_{m,i-1} \to x \to a_{m,i-2}$ in A_m. Moreover, $x \notin \{a_{m,1}, \ldots, a_{m,i-1}\}$. But there is only one element x in A_m with these properties, namely, $x = a_{m,i}$. Hence $\alpha(a_{n,i}) = a_{m,i}$. $\square$

Lemma 3.12. $\mathbf{A}_n$ *is a simple tournament for $n \geq 8$.*

Proof. Let $\vartheta \neq \mathrm{id}_{A_n}$ be a congruence of $\mathbf{A}_n$. We need to prove that $\vartheta = A_n \times A_n$.

If $(a_{n,i}, a_{n,i+1}) \in \vartheta$ for some i, then in the case $i > 1$ we have $a_{n,i-1} \to a_{n,i} \to a_{n,i+1} \to a_{n,i-1}$, from which it follows that $(a_{n,i-1}, a_{n,i}) \in \vartheta$; and in the case $i + 1 < n$ we have $(a_{n,i+1}, a_{n,i+2}) \in \vartheta$ for the same reason. Hence, if $(a_{n,i}, a_{n,i+1}) \in \vartheta$ for some i, then $\vartheta = A_n \times A_n$.

If $(a_{n,i}, a_{n,i+2}) \in \vartheta$ for some i, then $(a_{n,i}, a_{n,i+1}) = (a_{n,i}a_{n,i+1}, a_{n,i+2}a_{n,i+1}) \in \vartheta$, so that $\vartheta = A_n \times A_n$.

If $(a_{n,i}, a_{n,i+3}) \in \vartheta$ for some i, then one of the following two cases takes place. If $i \geq 3$, then $(a_{n,i}, a_{n,i-2}) = (a_{n,i}a_{n,i-2}, a_{n,i+3}a_{n,i-2}) \in \vartheta$. If $i \leq n-5$, then $(a_{n,i}, a_{n,i+5}) = (a_{n,i}a_{n,i+5}, a_{n,i+3}a_{n,i+5}) \in \vartheta$ and hence $(a_{n,i+3}, a_{n,i+5}) \in \vartheta$. But then, $\vartheta = A_n \times A_n$ in both cases.

Finally, if $(a_{n,i}, a_{n,j}) \in \vartheta$ and $j \geq i+4$, then $(a_{n,i}, a_{n,i+1}) = (a_{n,i}a_{n,i+1}, a_{n,j}a_{n,i+1}) \in \vartheta$, so that $\vartheta = A_n \times A_n$. □

Theorem 3.13. *The tournaments $\mathbf{A}_n$ with $n \geq 8$ are all simple and pairwise incomparable in the sense that if $n \neq m$, then $\mathbf{A}_n$ cannot be embedded into $\mathbf{A}_m$.*

Proof. It follows from the Lemmas 3.8–3.12. □

CHAPTER IV

STRONGLY CONNECTED ALGEBRAS

The compatible quasiordering

Definition 4.1. For an algebra $\mathbf{A} \in \mathcal{T}^3$ and two elements $a, b \in A$, write $a \lesssim b$ if there exist elements $a_0, \ldots, a_{k-1} \in A$ such that $a = a_0 \to^{\mathbf{A}} a_1 \to^{\mathbf{A}} \cdots \to^{\mathbf{A}} a_{k-1} = b$. Write $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$. Clearly, $\lesssim$ is a quasiordering and $\sim$ is an equivalence on A.

Lemma 4.2. *Let $\mathbf{A} \in \mathcal{T}^3$. Then $\lesssim$ is a compatible quasiordering, $\sim$ is a congruence of $\mathbf{A}$, and the factor $\mathbf{A}/\sim$ is a semilattice; actually, $\sim$ is precisely the least congruence of $\mathbf{A}$ such that the factor is a semilattice.*

Proof. Compatibility means that $a \lesssim b$ implies $ac \lesssim bc$; for this, it is sufficient to prove that $a \to b$ implies $ac \lesssim bc$. If $ab = a$, then $ac = aca = abca = baca = bcac \to bca \to bc$.

Consequently, $\sim$ is a congruence. Due to the equation (18) of Theorem 3.3, the factor $\mathbf{A}/\sim$ satisfies $xy \cdot z = xz \cdot y$; together with commutativity, this implies associativity. We have proved that $\mathbf{A}/\sim$ is a semilattice. Clearly, every congruence, the factor by which is a semilattice, contains $\sim$. $\square$

Subalgebras of subdirectly irreducibles

Lemma 4.3. *Let $\mathbf{A} \in \mathcal{T}^3$ be a subdirectly irreducible algebra, and $\mathbf{B}$ be a subalgebra of $\mathbf{A}$ such that $|B| \geq 2$ and $B^2 \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$. Then $\mathbf{B}$ is subdirectly irreducible.*

Proof. We argue that for every $\vartheta \in \operatorname{Con} \mathbf{B}$, $\vartheta \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$. Then it will follow that $\operatorname{Con} \mathbf{B}$ has a unique co-atom, and thus $\mathbf{B}$ is subdirectly irreducible.

Let $\vartheta \in \operatorname{Con} \mathbf{B}$ and take a pair $\langle a, b\rangle \in \vartheta$ of elements and $c \in A$. We need to show that $\langle ac, bc\rangle \in \vartheta \cup \mathrm{id}_A$. If $ac \notin B$ or $bc \notin B$, then $ac = bc$, because $B^2 \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$. So we can assume that $ac, bc \in B$. Then $\langle a \cdot ac, b \cdot ac\rangle \in \vartheta$, $\langle b \cdot (ac)(bc), a \cdot (ac)(bc)\rangle \in \vartheta$ and $\langle a \cdot bc, b \cdot bc\rangle \in \vartheta$, because ϑ is a congruence of $\mathbf{B}$. But $ac = a \cdot ac$, $b \cdot ac = b \cdot (ac)(bc)$, $a \cdot (ac)(bc) = a \cdot bc$ and $b \cdot bc = bc$, by Theorem 3.3 (3) and (8). Thus $\langle ac, bc\rangle \in \vartheta$. ☐

Corollary 4.4. *Let $\mathbf{A} \in \mathcal{T}^3$ be a subdirectly irreducible algebra, and $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. If $|B| \geq 2$ and B is a down-set of $\lesssim$, that is, $ba \in B$ for all $b \in B$ and $a \in A$, then $B^2 \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$ and therefore $\mathbf{B}$ is subdirectly irreducible.* ☐

Definition 4.5. Let $\mathbf{A} \in \mathcal{T}^3$. An element $0 \in A$ is the *zero element* of $\mathbf{A}$ if $0 \to a$ for all $a \in A$. An element $1 \in A$ is the *unit element* of $\mathbf{A}$ if $a \to 1$ for all $a \in A$. Clearly, $\mathbf{A}$ has at most one zero and one unit element.

Lemma 4.6. *Let $\mathbf{A} \in \mathcal{T}^3$ be a subdirectly irreducible algebra such that $|A| \geq 3$. If $\mathbf{A}$ has a zero element $0 \in A$, then $\mathbf{A} \setminus \{0\}$ is a subdirectly irreducible subalgebra of $\mathbf{A}$. If $\mathbf{A}$ has a unit element $1 \in A$, then $\mathbf{A} \setminus \{1\}$ is a subdirectly irreducible subalgebra of $\mathbf{A}$.*

Proof. Let $\mathbf{A} \in \mathcal{T}^3$ be a subdirectly irreducible algebra with a zero element $0 \in A$. We argue that $A \setminus \{0\}$ is a subuniverse of $\mathbf{A}$. To get a contradiction, suppose that $bc = 0$ for some elements $b, c \in A \setminus \{0\}$. Put $B = \{\, x \in A : x \lesssim b \,\}$ and $C = \{\, x \in A : x \lesssim c \,\}$. Clearly, $\{0\}$ is the least block of $\sim$, and $0 \in B \cap C$. On the other hand, if $x \in B \cap C$, then $x \lesssim b$ and $x \lesssim c$, so $x \lesssim bc = 0$. Therefore $B \cap C = \{0\}$. Then, by Corollary 4.4, $B^2 \cup \mathrm{id}_A$ and $C^2 \cup \mathrm{id}_A$ are nontrivial congruences of $\mathbf{A}$, and their intersection is trivial. This contradicts the assumption that $\mathbf{A}$ is subdirectly irreducible. Hence $\mathbf{A} \setminus \{0\}$ is a subalgebra of $\mathbf{A}$. Now it is clear that $(A \setminus \{0\})^2 \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$, therefore $\mathbf{A} \setminus \{0\}$ is subdirectly irreducible by Lemma 4.3.

If $\mathbf{A}$ has a unit element $1 \in A$, then, clearly, $\mathbf{A} \setminus \{1\}$ is a subalgebra of $\mathbf{A}$ and $(A \setminus \{1\})^2 \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$. Therefore $\mathbf{A} \setminus \{1\}$ is subdirectly irreducible. □

Reduction to strongly connected subdirectly irreducibles

Lemma 4.7. *Let $\mathbf{A} \in \mathcal{T}^3$ be such that the least block B of $\sim$ is a tournament. Then for every element $a \in A \setminus B$, such that a is incomparable with at least one element of B, there exists a unique element $a' \in B$ with the following two properties:*

(1) *$ax = a'$ for any $x \in B$ incomparable with a (in particular, $a' \to a$);*

(2) *$y \to a'$ for any $y \in B$ such that $y \to a$.*

Proof. Suppose $ax_1 \neq ax_2$ for two elements $x_1, x_2 \in B$ incomparable with a. We have either $ax_1 \to x_2$ or $x_2 \to ax_1$. If $ax_1 \to x_2$, then $ax_1 \to x_2$ and $ax_1 \to a$ imply $ax_1 \to ax_2$ by the properties of a product. If $x_2 \to ax_1$, then $x_2 \to ax_1 \to a$ implies $ax_1 \to ax_2$ by Lemma 3.4 (6). So, $ax_1 \to ax_2$ in any case. But then $ax_2 \to ax_1$ by symmetry, and we get $ax_1 = ax_2$.

Take an arbitrary element $x \in B$ which is incomparable with a, and put $a' = ax$. Let $y \in B$ be such that $y \to a$. The only alternative to $y \to a'$ could be $a' \to y$, so suppose that. Since $a' = ax$ and $a' \to y \to a$, we have $xy = a'$. But xy is either x or y, so $y = a'$. □

Lemma 4.8. *Let $\mathbf{A} \in \mathcal{T}^3$ be a finite, subdirectly irreducible algebra such that the least block B of $\sim$ is a tournament. Then $x \to a$ for any $x \in B$ and any $a \in A \setminus B$.*

Proof. Suppose, on the contrary, that some element of $A \setminus B$ is incomparable with at least one element of B, and take a minimal (with respect to $\lesssim$) such element a. Take an element $x \in B$ incomparable with a and put $a' = ax$. If there is an element b such that $B < b/{\sim} < a/{\sim}$, then there is one such element with $b \to a$ (replace b with ab if necessary); we have $x \to b$ by the minimality of a, so that $b \to a'$ by Lemma 3.4 (6), a contradiction. This proves that $a/{\sim}$ is an atom in $\mathbf{A}/{\sim}$. So by Corollary 4.4, it is sufficient to assume that $A = B \cup (a/{\sim})$. Thus $A \setminus B$ is a subuniverse of $\mathbf{A}$.

The set $A \setminus B$ can be partitioned into two subsets: the (possibly empty) subset C of the elements c satisfying $x \to c$ for all $x \in B$, and the subset D of the elements a for which the element $a' \in B$, as in Lemma 4.7, exists. Denote by θ the equivalence on A with blocks $\{x\} \cup \{\, a \in D : a' = x \,\}$ for $x \in B$ (and singletons, corresponding to the elements of C). The following three observations will imply that θ is a congruence of $\mathbf{A}$.

Claim 1. *If $a \in D$ and $b \in B$, then $ab \in B$ and $ab = a'b$.*

If a and b are incomparable, then $ab = a' = a'b$. The other possibility is that $b \to a$. Then $b \to a'$ and $ab = b = a'b$.

Claim 2. *If $a \in D$ and $b \in C$, then $ab \in D$ and $(ab)' = a'b = a'$.*

Since $a' \to a$ and $a' \to b$, we have $a' \to ab$. Since $a' \to ab \to a$, we have $x \cdot ab = xa = a'$ by Lemma 3.4 (3). Clearly, $ab \in A \setminus B$, thus either $x \to ab$, or x and ab are incomparable. If $x \to ab$, then $x \to ab \to a$ implies $ab \to a'$ by Lemma 3.4 (6), a contradiction.

Claim 3. *If $a, b \in D$ and $a' \to b'$, then $ab \in D$ and $(ab)' = a'b = ab' = a'$.*

Since $a' \to b'$, we have $b'a = a'$. By the definition of b', $a' \to b'$ implies $a' \to b$. By Theorem 3.3 (18) we get $ab \cdot b' = b'abab' = a'bab' = a'ab' = a'b' = a'$. Clearly, $ab \in A \setminus B$. It remains to prove that ab and b' are incomparable. If $b' \to ab$, then $b' \to ab \to a$ gives $ab \to a'$, a contradiction.

We conclude that θ is a congruence of $\mathbf{A}$. This gives us a contradiction with Corollary 4.4, since θ is nontrivial, $|B| \geq 2$, and $\theta \cap B^2 = \mathrm{id}$. $\square$

Lemma 4.9. *Let $\mathbf{A} \in \mathcal{T}^3$ be a finite, subdirectly irreducible algebra without zero, such that the least block of $\sim$ is a tournament. Then $\mathbf{A}$ is a tournament.*

Proof. First, the least block B of $\sim$ has at least two elements, since $\mathbf{A}$ has no zero. Suppose that $\mathbf{A}$ contains a pair of incomparable elements. By Lemma 4.8, both elements must belong to $A \setminus B$. If $A \setminus B$ is a subgroupoid, then $(A \setminus B)^2 \cup \mathrm{id}_A$ is a congruence, which is not possible. So, let $ab \in B$ for some $a, b \in A \setminus B$. For every $x \in B$ we have $x \to a$ and $x \to b$ and hence $x \to ab$. Thus ab is the unit element of $\mathbf{B}$, which contradicts that B is a block of $\sim$. $\square$

Definition 4.10. An algebra $\mathbf{A} \in \mathcal{T}^3$ is *strongly connected* if $\sim = A^2$, that is, for all pairs a, b of elements of A there exist elements $a_0, \ldots, a_{k-1} \in A$ such that $a = a_0 \to^{\mathbf{A}} a_1 \to^{\mathbf{A}} \cdots \to^{\mathbf{A}} a_{k-1} = b$.

Theorem 4.11. *Every finite, subdirectly irreducible algebra in $\mathcal{T}^3$ which is not a tournament contains a strongly connected, subdirectly irreducible subalgebra which is again not a tournament.*

Proof. By Lemma 4.6 we can assume that the algebra has no zero element. By Corollary 4.4 the least block of $\sim$ does the job, unless it is a tournament. However, it is not a tournament by Lemma 4.9. $\square$

CHAPTER V

SUBDIRECTLY IRREDUCIBLES

Subdirect products of strongly connected tournaments

Lemma 5.1. *Let $\mathbf{A} \in \mathcal{T}$ be a finite, strongly connected algebra which is a homomorphic image of a subalgebra $\mathbf{B}$ of a product of tournaments. Then there exists a finite subalgebra $\mathbf{C} \leq \mathbf{B}$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{C}$, and $\mathbf{C}$ is a subdirect product of finitely many strongly connected finite tournaments.*

Proof. We can assume that $\mathbf{B}$ is a finite subdirect product $\mathbf{B} \leq \prod_{i<k} \mathbf{T}_i$ of finitely many finite tournaments $\mathbf{T}_i$, by Theorem 3.1. Let φ be the homomorphism of $\mathbf{B}$ onto $\mathbf{A}$. Take a traversal $f : A \to B$ for φ, that is, a mapping such that $\varphi f(a) = a$ for all $a \in A$. Since $\mathbf{A}$ is finite and strongly connected, there exists a loop $a_0 \leftarrow^{\mathbf{A}} a_1 \leftarrow^{\mathbf{A}} \cdots \leftarrow^{\mathbf{A}} a_{n-1} \leftarrow^{\mathbf{A}} a_0$ that goes through (possibly more than once) all elements of $\mathbf{A}$. For an integer i, put $i' = i \bmod n$. We define an infinite sequence $b_0, b_1, \ldots \in B$ of elements by

$$b_0 = f(a_0), \text{ and}$$
$$b_i = b_{i-1} f(a_{i'}) \text{ for } i > 0.$$

Using induction it is easy to check that $\varphi(b_i) = a_{i'}$ for all $i \geq 0$. Indeed, $\varphi(b_0) = \varphi f(a_0) = a_0$, and

$$\begin{aligned} \varphi(b_i) &= \varphi(b_{i-1} \cdot f(a_{i'})) = \varphi(b_{i-1}) \cdot \varphi f(a_{i'}) \\ &= a_{(i-1)'} \cdot a_{i'} = a_{i'} \end{aligned}$$

for $i > 0$. In particular, $\varphi(b_{jn}) = a_0$ for all $j \geq 0$. Since $\mathbf{B}$ is finite, there must exist a pair of positive integers $s < t$ such that $b_{sn} = b_{tn}$. Notice that

$$b_{sn} \leftarrow^{\mathbf{B}} b_{sn+1} \leftarrow^{\mathbf{B}} \cdots \leftarrow^{\mathbf{B}} b_{tn-1} \leftarrow^{\mathbf{B}} b_{sn}. \tag{$*$}$$

Put $C_0 = \{b_{sn}, b_{sn+1}, \ldots, b_{tn-1}\}$, and let $\mathbf{C}$ be the subalgebra of $\mathbf{B}$ generated by C_0. Clearly, $\varphi(C) = A$, because $A \supseteq \varphi(C) \supseteq \varphi(C_0) = \{\varphi(b_{sn}), \ldots, \varphi(b_{tn-1})\} = \{a_0, \ldots, a_{n-1}\} = A$. Therefore $\mathbf{A}$ is a homomorphic image of $\mathbf{C}$. We claim that $\mathbf{C}$ is a subdirect product of strongly connected tournaments. Put $S_i = \pi_i(C_0)$ for all $i < k$. Since all nonempty subsets of T_i are subuniverses, $\mathbf{S}_i$ is a subtournament of $\mathbf{T}_i$. Therefore, $\pi_i(C) = \pi_i(C_0) = S_i$. On the other hand, $\pi_i(C_0) = \{\pi_i(b_{sn}), \pi_i(b_{sn+1}), \ldots, \pi_i(b_{tn-1})\}$. But by $(*)$,

$$\pi_i(b_{sn}) \leftarrow^{\mathbf{T}_i} \pi_i(b_{sn+1}) \leftarrow^{\mathbf{T}_i} \ldots \pi_i(b_{tn-1}) \leftarrow^{\mathbf{T}_i} \pi_i(b_{sn}),$$

hence $\mathbf{S}_i$ is strongly connected for all $i < k$. □

Lemma 5.2. *Let $\mathbf{A}$ be a subdirect product of two strongly connected algebras $\mathbf{B}, \mathbf{C} \in \mathcal{T}$, and suppose that $B \times \{c\} \subseteq A$ for some element $c \in C$. Then $A = B \times C$.*

Proof. Clearly, it is enough to show that $B \times \{c'\} \subseteq A$ for all $c' \in C$. Since $\mathbf{C}$ is strongly connected, for any element $c' \in C$ there exists a path $c = c_0 \leftarrow^{\mathbf{C}} c_1 \leftarrow^{\mathbf{C}} \cdots \leftarrow^{\mathbf{C}} c_{k-1} = c'$. Then it is enough to show that $B \times \{c_i\} \subseteq A$ implies $B \times \{c_{i+1}\} \subseteq A$. Since $\mathbf{A}$ is a subdirect product of $\mathbf{B}$ and $\mathbf{C}$, there must exist an element $b \in B$ such that $\langle b, c_{i+1}\rangle \in A$. We want to show that $\langle b', c_{i+1}\rangle \in A$ for all $b' \in B$. Since $\mathbf{B}$ is strongly connected, there is a path $b = b_0 \leftarrow^{\mathbf{B}} b_1 \leftarrow^{\mathbf{B}} \cdots \leftarrow^{\mathbf{B}} b_{t-1} = b'$ connecting b and b'. Notice that $\langle b_0, c_i\rangle, \langle b_1, c_i\rangle, \ldots, \langle b_{t-1}, c_i\rangle \in A$ because we have assumed that $B \times \{c_i\} \subseteq A$. The product $\langle b, c_{i+1}\rangle\langle b_0, c_i\rangle\langle b_1, c_i\rangle \ldots \langle b_{t-1}, c_i\rangle$ of elements of $\mathbf{A}$ is the pair $\langle b', c_{i+1}\rangle$. Hence $\langle b', c_{i+1}\rangle \in A$. □

Lemma 5.3. *Let $\mathbf{A}$ and $\mathbf{B}$ be strongly connected algebras in $\mathcal{T}$. Then* $\mathrm{Con}(\mathbf{A} \times \mathbf{B}) \cong$

$\operatorname{Con}\mathbf{A} \times \operatorname{Con}\mathbf{B}$.

Proof. Put $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. We need to prove that for each congruence $\vartheta \in \operatorname{Con}\mathbf{C}$ there exist congruences $\alpha \in \operatorname{Con}\mathbf{A}$ and $\beta \in \operatorname{Con}\mathbf{B}$ so that $\vartheta = \alpha \times \beta$. Clearly, every congruence of $\mathbf{C}$ is a join of principal congruences $\operatorname{Cg}_{\mathbf{C}}(c, c')$ where $c \to^{\mathbf{C}} c'$. Because the join of product congruences is also a product congruence, it is enough to show the existence of α and β in the case when $\vartheta = \operatorname{Cg}_{\mathbf{C}}(c, c')$ and $c \to^{\mathbf{C}} c'$.

Put $c = \langle a, b\rangle$ and $c' = \langle a', b'\rangle$ for elements $a, a' \in A$ and $b, b' \in B$. Clearly $a \to^{\mathbf{A}} a'$ and $b \to^{\mathbf{B}} b'$. Define $\alpha = \operatorname{Cg}_{\mathbf{A}}(a, a')$ and $\beta = \operatorname{Cg}_{\mathbf{B}}(b, b')$.

Now we argue that $\alpha \times \mathrm{id}_B \subseteq \vartheta$ where id_B is the diagonal relation on B. We will show that $\alpha \times \{\langle b'', b''\rangle\} \subseteq \vartheta$ for all $b'' \in B$. Since $\mathbf{B}$ is strongly connected, there exists a path $b = b_0 \leftarrow^{\mathbf{B}} b_1 \leftarrow^{\mathbf{B}} \cdots \leftarrow^{\mathbf{B}} b_{k-1} = b''$ connecting b and b''. Observe that the unary polynomial $q(x) = x\langle a', b_0\rangle\langle a', b_1\rangle \ldots \langle a', b_{k-1}\rangle$ of $\mathbf{C}$ maps c to $\langle a, b''\rangle$ and c' to $\langle a', b''\rangle$. Thus $\langle a, b''\rangle \vartheta \langle a', b''\rangle$. But this implies that $\alpha \times \{\langle b'', b''\rangle\} \subseteq \vartheta$.

By a similar argument we get that $\mathrm{id}_A \times \beta \subseteq \vartheta$. Then $\alpha \times \beta = (\alpha \times \mathrm{id}_B) \vee (\mathrm{id}_A \times \beta) \subseteq \vartheta$. On the other hand, $\langle c, c'\rangle \in \alpha \times \beta$, thus $\vartheta \subseteq \alpha \times \beta$. □

Triangular graphs

Definition 5.4. We define the class of *triangular graphs* inductively in the following way. All triangles, directed graphs on a set $\{a, b, c\}$ with edges $a \to b \to c \to a$, are triangular. Now given two triangular graphs $\mathbf{G}$ and $\mathbf{H}$, such that $\mathbf{G}$ and $\mathbf{H}$ have at least one common edge, and no edge $x \to^{\mathbf{G}} y$ such that $x \leftarrow^{\mathbf{H}} y$, then the directed graph $\langle G \cup H; \to^{\mathbf{G}} \cup \to^{\mathbf{H}}\rangle$ is triangular, as well.

It is worth noting a few basic properties of triangular graphs which follow immediately from the definition. Let $\mathbf{G}$ be a triangular graph. Then $\mathbf{G}$ is finite, and has no loops, that is, a vertex $a \in G$ such that $a \to^{\mathbf{G}} a$. Moreover, $\mathbf{G}$ has no edge $a \to^{\mathbf{G}} b$ such that $a \leftarrow^{\mathbf{G}} b$. Furthermore, $\mathbf{G}$ is a union of triangles, and is strongly connected.

Lemma 5.5. *Let $\mathbf{G}$ be a triangular graph and f be a map of a set H onto G. Then the graph $\mathbf{H} = \langle H; f^{-1}(\to^{\mathbf{G}})\rangle$, with edges $x \to^{\mathbf{H}} y$ if and only if $f(x) \to^{\mathbf{G}} f(y)$, is triangular, as well.*

Proof. Take a traversal set $H_0 \subseteq H$ of f, that is, $|f^{-1}(x) \cap H_0| = 1$ for all $x \in G$. Clearly, the graph $\mathbf{H}_0 = \mathbf{H}|_{H_0}$ is isomorphic to $\mathbf{G}$, and therefore triangular. Now for each edge $a \to^{\mathbf{H}} b$ we find a set $H_{a,b} \subseteq H$ such that the graph $\mathbf{H}_{a,b} = \mathbf{H}|_{H_{a,b}}$ is triangular, contains the edge $a \to b$ and has at least one common edge with $\mathbf{H}_0$. Then the graph $\mathbf{H} = \mathbf{H}_0 \cup \bigcup\{\, \mathbf{H}_{a,b} : a \to^{\mathbf{H}} b\,\}$ is triangular, by definition.

Since $a \to^{\mathbf{H}} b$, $f(a) \to^{\mathbf{G}} f(b)$. Every edge of $\mathbf{G}$ is an edge of a triangle, that is, there is an element $c \in G$ such that $c \to^{\mathbf{G}} f(a) \to^{\mathbf{G}} f(b) \to^{\mathbf{G}} c$. Take elements $a_0, b_0, c_0 \in H_0$ such that $f(a_0) = f(a)$, $f(b_0) = f(b)$ and $f(c_0) = c$. Finally, define $H_{a,b} = \{a, b, a_0, b_0, c_0\}$. Clearly, the triangle $a_0 \to^{\mathbf{H}_{a,b}} b_0 \to^{\mathbf{H}_{a,b}} c_0 \to^{\mathbf{H}_{a,b}} a_0$ has all its edges common with $\mathbf{H}_0$. However, it also has a common edge with $a \to^{\mathbf{H}_{a,b}} b_0 \to^{\mathbf{H}_{a,b}} c_0 \to^{\mathbf{H}_{a,b}} a$, which has a common edge with $a \to^{\mathbf{H}_{a,b}} b \to^{\mathbf{H}_{a,b}} c_0 \to^{\mathbf{H}_{a,b}} a$, which in turn has a common edge with $a_0 \to^{\mathbf{H}_{a,b}} b_0 \to^{\mathbf{H}_{a,b}} c_0 \to^{\mathbf{H}_{a,b}} a_0$. Observe that $\mathbf{H}_{a,b}$ is a union of these four triangles, hence $\mathbf{H}_{a,b}$ is triangular. $\square$

For a set G, we denote by $\mathbf{F}(G)$ the free algebra in $\mathcal{T}$ freely generated by G.

Lemma 5.6. *Let $\mathbf{G}$ be a triangular graph. Then there exists an endomorphism τ of the free algebra $\mathbf{F}(G)$ that satisfies the following two statements for all tournaments $\mathbf{T}$ and homomorphisms $\varphi : \mathbf{F}(G) \to \mathbf{T}$.*

(1) *If $\varphi(x) \to^{\mathbf{T}} \varphi(y)$ for all edges $x \to^{\mathbf{G}} y$, then $\varphi\tau = \varphi$.*

(2) *If $\varphi(x) \leftarrow^{\mathbf{T}} \varphi(y)$ for some edge $x \to^{\mathbf{G}} y$, then $\varphi\tau$ is constant.*

Proof. We prove the lemma by induction on the complexity of the triangular graph $\mathbf{G}$. In addition to (1) and (2) we will also prove the following statement.

(3) If $\{\, \varphi(x) : x \in G \,\} = \{s, t\}$ for some elements $s, t \in T$, then $\varphi\tau$ is the constant st-valued homomorphism.

Notice that it is enough to check the conclusions of statements (1) – (3) on the generating set G of $\mathbf{F}(G)$, because $\varphi\tau$ is a homomorphism.

CLAIM 1. *Let $\mathbf{G}$ be the 3-element triangular graph on a set $\{a, b, c\}$ with edges $a \rightarrow^{\mathbf{G}} b \rightarrow^{\mathbf{G}} c \rightarrow^{\mathbf{G}} a$. Let τ be the endomorphism of $\mathbf{F}(G)$ defined by*

$$\tau(c) = cbac,$$

$$\tau(b) = cbac(cb) = \tau(c)(cb),$$

$$\tau(a) = cbac(cb)(cba) = \tau(b)(cba).$$

Then τ satisfies statements (1) – (3).

First we check statement (2). If $\varphi(b) \leftarrow^{\mathbf{T}} \varphi(c)$, then

$$\varphi\tau(c) = \varphi(c)\varphi(b)\varphi(a)\varphi(c) = \varphi(c)\varphi(a)\varphi(c) = \varphi(c)\varphi(a),$$

$$\varphi\tau(b) = \varphi\tau(c)\varphi(cb) = \varphi(c)\varphi(a)\varphi(c) = \varphi(c)\varphi(a),$$

$$\varphi\tau(a) = \varphi\tau(b)\varphi(cba) = \varphi(c)\varphi(a)(\varphi(c)\varphi(a)) = \varphi(c)\varphi(a).$$

Thus $\varphi\tau$ is constant as claimed. So we can assume that $\varphi(b) \rightarrow^{\mathbf{T}} \varphi(c)$. If $\varphi(a) \leftarrow^{\mathbf{T}} \varphi(b)$, then

$$\varphi\tau(c) = \varphi(c)\varphi(b)\varphi(a)\varphi(c) = \varphi(b)\varphi(a)\varphi(c) = \varphi(b)\varphi(c),$$

$$\varphi\tau(b) = \varphi\tau(c)\varphi(cb) = \varphi(b)\varphi(c)\varphi(b) = \varphi(b)\varphi(c),$$

$$\varphi\tau(a) = \varphi\tau(b)\varphi(cba) = \varphi(b)\varphi(c)\varphi(b) = \varphi(b)\varphi(c).$$

Thus $\varphi\tau$ is constant in this case, as well. So we can also assume that $\varphi(a) \to^{\mathbf{T}} \varphi(b)$. Finally, if $\varphi(c) \leftarrow^{\mathbf{T}} \varphi(a)$, then

$$\begin{aligned}
\varphi\tau(c) &= \varphi(c)\varphi(b)\varphi(a)\varphi(c) = \varphi(b)\varphi(a)\varphi(c) = \varphi(a)\varphi(c) = \varphi(a), \\
\varphi\tau(b) &= \varphi\tau(c)\varphi(cb) = \varphi(a)\varphi(c)\varphi(b) = \varphi(a)\varphi(b) = \varphi(a), \\
\varphi\tau(a) &= \varphi\tau(b)\varphi(cba) = \varphi(a)\varphi(c)\varphi(b)\varphi(a) = \varphi(a).
\end{aligned}$$

Thus $\varphi\tau$ is constant, once again. This proves that $\varphi\tau$ is constant whenever $\varphi(x) \leftarrow^{\mathbf{T}} \varphi(y)$ for some edge $x \to^{\mathbf{G}} y$.

The other alternative is that $\varphi(x) \to^{\mathbf{T}} \varphi(y)$ for all edges $x \to^{\mathbf{G}} y$. Then $\varphi\tau = \varphi$, because

$$\begin{aligned}
\varphi\tau(c) &= \varphi(c)\varphi(b)\varphi(a)\varphi(c) = \varphi(c), \\
\varphi\tau(b) &= \varphi\tau(c)\varphi(cb) = \varphi(c)\varphi(b) = \varphi(b), \\
\varphi\tau(a) &= \varphi\tau(b)\varphi(cba) = \varphi(b)\varphi(a) = \varphi(a).
\end{aligned}$$

Finally, statement (3) holds, because all variables a, b and c occur in $\tau(a), \tau(b)$ and $\tau(c)$.

Claim 2. *Let $\mathbf{G}$ be the union of two triangular graphs $\mathbf{G}_0$ and $\mathbf{G}_1$ with a common edge $a \to b$, $a, b \in G_0 \cap G_1$. By the induction hypothesis, there are endomorphisms $\hat{\tau}_i$ $(i = 0, 1)$ of $\mathbf{F}(G_i)$ satisfying statements $(1) - (3)$ for $\mathbf{G}_i$. Let τ_i be the endomorphism of $\mathbf{F}(G)$ defined by*

$$\tau_i(x) = \begin{cases} \hat{\tau}_i(x) & \text{if } x \in G_i, \\ x & \text{if } x \in G \setminus G_i. \end{cases}$$

Then $\tau = \tau_0\tau_1\tau_0\tau_1\tau_0$ satisfies statements $(1) - (3)$ for $\mathbf{G}$.

Let φ be a homomorphism of $\mathbf{F}(G)$ into a tournament $\mathbf{T}$. Notice that $\hat{\tau}_i = \tau_i|_{F(G_i)}$

for $i = 0, 1$, by definition. To check statement (1), assume that $\varphi(x) \to^{\mathbf{T}} \varphi(y)$ for all edges $x \to^{\mathbf{G}} y$. Then, by the induction hypothesis, $\varphi|_{F(G_i)}\hat{\tau}_i(x) = \varphi|_{F(G_i)}(x)$ for all $x \in G_i$, $i = 0, 1$. Thus $\varphi\tau_i(x) = \varphi(x)$ for all $x \in G$, and therefore $\varphi\tau_i = \varphi$. Hence $\varphi\tau = \varphi\tau_0\tau_1\tau_0\tau_1\tau_0 = \varphi\tau_1\tau_0\tau_1\tau_0 = \cdots = \varphi$, as claimed.

To check statement (2), first assume that $\varphi(x) \to^{\mathbf{T}} \varphi(y)$ for all edges $x \to^{\mathbf{G}_0} y$, but $\varphi(x) \leftarrow^{\mathbf{T}} \varphi(y)$ for some edge $x \to^{\mathbf{G}_1} y$. By the same argument as above, $\varphi\tau_0 = \varphi$, so we need to show that $\varphi\tau_1\tau_0\tau_1\tau_0$ is constant. By the induction hypothesis, $\varphi|_{F(G_1)}\hat{\tau}_1$ is a constant s-valued homomorphism of $\mathbf{F}(G_1)$ for some element $s \in T$. In particular, $\varphi\tau_1(a) \leftarrow^{\mathbf{T}} \varphi\tau_1(b)$ for the common edge $a \to^{\mathbf{G}_0} b$. Now we can apply the induction hypothesis again for the homomorphism $\varphi\tau_1|_{F(G_0)}$ and endomorphism $\hat{\tau}_0$ of $\mathbf{F}(G_0)$, and obtain that $\varphi\tau_1|_{F(G_0)}\hat{\tau}_0$ is a constant t-valued homomorphism of $\mathbf{F}(G_0)$ for some $t \in T$. We argue that $\{\, \varphi\tau_1\tau_0(x) : x \in G \,\} = \{s, t\}$. For $x \in G_0$, $\varphi\tau_1\tau_0(x) = \varphi\tau_1|_{F(G_0)}\hat{\tau}_0(x) = t$. On the other hand, $\varphi\tau_1\tau_0(x) = \varphi\tau_1(x) = \varphi|_{F(G_1)}\hat{\tau}_1(x) = s$ for all $x \in G \setminus G_0$. By statement (3) of the induction hypothesis, $\varphi\tau_1\tau_0|_{F(G_1)}\hat{\tau}_1$ is the constant st-valued homomorphism of $\mathbf{F}(G_1)$. Thus $\varphi\tau_1\tau_0\tau_1(x) = st$ for all $x \in G_1$, and $\varphi\tau_1\tau_0\tau_1(x) = t$ for all $x \in G \setminus G_1$. Hence $\{\, \varphi\tau_1\tau_0\tau_1(x) : x \in G \,\} = \{st, t\}$. Applying the induction hypothesis, for the last time, for the homomorphism $\varphi\tau_1\tau_0\tau_1|_{F(G_0)}$ and endomorphism $\hat{\tau}_0$, we conclude that $\varphi\tau_1\tau_0\tau_1\tau_0(x) = st$ for all $x \in G$. This finishes the proof of statement (2) when $\varphi(x) \to^{\mathbf{T}} \varphi(y)$ for all edges $x \to^{\mathbf{G}_0} y$, but $\varphi(x) \leftarrow^{\mathbf{T}} \varphi(y)$ for some edge $x \to^{\mathbf{G}_1} y$. The case when $\varphi(x) \leftarrow^{\mathbf{T}} \varphi(y)$ for some edge $x \to^{\mathbf{G}_1} y$ is proved similarly. But this time we get that $\varphi\tau_0\tau_1\tau_0\tau_1$ is already constant.

Finally, τ clearly satisfies statement (3). $\square$

Definition 5.7. We say that an algebra $\mathbf{A} \in \mathcal{T}$ has a *spanning triangular graph* $\mathbf{G}$, if $\mathbf{G}$ is triangular, defined on the whole set A, and $\to^{\mathbf{G}} \subseteq \to^{\mathbf{A}}$.

Lemma 5.8. *Every finite, nontrivial strongly connected tournament has a spanning triangular graph.*

Proof. Let $\mathbf{T}$ be a finite, nontrivial strongly connected tournament. We prove the lemma by induction on the number of elements of $\mathbf{T}$. The smallest nontrivial strongly connected tournament is the three element cycle, which has a spanning triangular graph.

Choose an element $a \in T$ and consider the tournament $\mathbf{T} \setminus \{a\}$. Let $C_0, \ldots, C_{k-1}$ be the strongly connected components of $\mathbf{T} \setminus \{a\}$. Clearly, for each pair of indices $0 \leq i < j < k$, either $b \to^{\mathbf{T}} c$ for all $b \in C_i$ and $c \in C_j$, or $c \to^{\mathbf{T}} b$ for all $b \in C_i$ and $c \in C_j$. We will use the notation $C_i \to^{\mathbf{T}} C_j$ in the former, and $C_j \to^{\mathbf{T}} C_i$ in the latter case. This defines a k-element tournament on the set $\{C_0, \ldots, C_{k-1}\}$. Notice that this tournament must be a chain. Without loss of generality, we can assume that $C_0 \to^{\mathbf{T}} \cdots \to^{\mathbf{T}} C_{k-1}$.

Assume that $k = 1$. Then, the tournament $\mathbf{C}_0$ is nontrivial, and by the induction hypothesis, $\mathbf{C}_0$ has a spanning triangular graph $\mathbf{G}$. Since $\mathbf{T}$ is strongly connected, there are elements $b, c \in C_0$ such that $c \to^{\mathbf{T}} a \to^{\mathbf{T}} b$. Let $b = b_0 \to^{\mathbf{G}} b_1 \to^{\mathbf{G}} \cdots \to^{\mathbf{G}} b_{t-1} = c$ be a path in $\mathbf{C}_0$ with edges from $\mathbf{G}$. Since $a \to^{\mathbf{T}} b_0$ and $b_{t-1} \to^{\mathbf{T}} a$, there exists an index $j < t-1$ such that $a \to^{\mathbf{T}} b_j \to^{\mathbf{G}} b_{j+1} \to^{\mathbf{T}} a$. Thus the graph $\langle T; \to^{\mathbf{G}} \cup \{a \to b_j \to b_{j+1} \to a\}\rangle$ is a spanning triangular graph of $\mathbf{T}$.

Now assume that $k > 1$. Since $\mathbf{T}$ is strongly connected, there exist elements $b \in C_0$ and $c \in C_{k-1}$ such that $c \to^{\mathbf{T}} a \to^{\mathbf{T}} b$. Since $C_0 \to^{\mathbf{T}} C_{k-1}$, we have $b \to^{\mathbf{T}} c$, as well. We are going to build a spanning triangular graph of $\mathbf{T}$ by finding triangular subgraphs of $\to^{\mathbf{T}}$ which have a common edge with the triangle $a \to^{\mathbf{T}} b \to^{\mathbf{T}} c \to^{\mathbf{T}} a$, and cover all elements of $T \setminus \{a\}$. For the elements $d \in C_1 \cup \cdots \cup C_{k-2}$ we can take the triangles $a \to^{\mathbf{T}} b \to^{\mathbf{T}} d \to^{\mathbf{T}} a$ or $a \to^{\mathbf{T}} d \to^{\mathbf{T}} c \to^{\mathbf{T}} a$.

For the elements of C_0, we consider two cases. If $a \to^{\mathbf{T}} d$ for all $d \in C_0$, then we can take the triangles $a \to^{\mathbf{T}} d \to^{\mathbf{T}} c \to^{\mathbf{T}} a$, once again. If $d \to^{\mathbf{T}} a$ for some $d \in C_0$, then we can argue similarly to the case $k = 1$, as follows. By the induction hypothesis, $\mathbf{C}_0$ has a spanning triangular graph $\mathbf{G}$. Now we can find a path $b = b_0 \to^{\mathbf{G}} \cdots \to^{\mathbf{G}} b_{t-1} = d$

in $\mathbf{C}_0$. Since $a \to^{\mathbf{T}} b_0$ and $b_{t-1} \to^{\mathbf{T}} a$, there exists an index $j < t-1$ such that $a \to^{\mathbf{T}} b_j \to^{\mathbf{T}} b_{j+1} \to^{\mathbf{T}} a$. Then the graph

$$\langle C_0 \cup \{a, c\}; \to^{\mathbf{G}} \cup \{a \to b_j \to b_{j+1} \to a\} \cup \{a \to b_j \to c \to a\}\rangle$$

is a triangular subgraph of $\to^{\mathbf{T}}$, have a common edge with $a \to^{\mathbf{T}} b \to^{\mathbf{T}} c \to^{\mathbf{T}} a$, and covers all elements of C_0.

Finally, a similar argument works for the elements of C_{k-1}, which completes the proof. $\square$

Weakly indecomposable subdirect products

Definition 5.9. We call a subdirect product $\mathbf{A} \leq \prod_{i \in I} \mathbf{A}_i$ of algebras *weakly indecomposable* if whenever $K \cup L$ is a partition of I such that $A = \pi_K(A) \times \pi_L(A)$, then $|\pi_K(A)| = 1$ or $|\pi_L(A)| = 1$.

Lemma 5.10. *Let $\mathbf{A}$ be a nontrivial, weakly indecomposable subdirect product of finitely many strongly connected finite tournaments. Then $\mathbf{A}$ has a maximal spanning triangular graph $\mathbf{G}$ such that for all elements $x \to^{\mathbf{A}} y \to^{\mathbf{A}} z$, if $x \to^{\mathbf{G}} y$ or $y \to^{\mathbf{G}} z$, then x and z are comparable in $\mathbf{A}$. Consequently, every subdirect product of finitely many strongly connected finite tournaments is strongly connected.*

Proof. Let $\mathbf{A} \leq \prod_{i<k} \mathbf{T}_i$ be a nontrivial, weakly indecomposable subdirect product of finitely many strongly connected finite tournaments $\mathbf{T}_i$. We are going to prove the statements of the lemma by a simultaneous induction on k. In particular, assume that all subdirect products of less than k finite, strongly connected tournaments are strongly connected. Notice that the base of the induction trivially holds.

CLAIM 1. *Let* $\mathbf{G}$ *be any triangular graph on* A*, and define* $K = \{\, i < k : \pi_i(x) \to^{\mathbf{T}_i} \pi_i(y)$ *for all edges* $x \to^{\mathbf{G}} y \,\}$*. If* $|\pi_K(A)| > 1$*, then* $x \to^{\mathbf{A}} y$ *for all edges* $x \to^{\mathbf{G}} y$*.*

Put $L = \{0, 1, \ldots, k-1\} \setminus K$. By definition, for all $i \in L$ there exists an edge $x \to^{\mathbf{G}} y$ such that $\pi_i(x) \leftarrow^{\mathbf{T}_i} \pi_i(y)$ and $\pi_i(x) \neq \pi_i(y)$. Notice that if $|\pi_L(A)| = 1$, then $L = \emptyset$ and $x \to^{\mathbf{A}} y$ for all edges $x \to^{\mathbf{G}} y$. So, to get a contradiction, assume that $|\pi_K(A)| > 1$ and $|\pi_L(A)| > 1$. Let $\mathbf{F}(A)$ be the free algebra in $\mathcal{T}$ freely generated by A, ι be the homomorphism of $\mathbf{F}(A)$ onto $\mathbf{A}$ extending the identity map of A, and τ be the endomorphism of $\mathbf{F}(A)$ satisfying the statements of Lemma 5.6 for the graph $\mathbf{G}$. Hence $\pi_i\iota\tau = \pi_i\iota$ for all $i \in K$, and $\pi_i\iota\tau$ is constant for all $i \in L$. Therefore $\pi_K\iota\tau = \pi_K\iota$ and $\pi_L\iota$ is constant. Now put $\mathbf{B} = \iota\tau(\mathbf{F}(A))$, which is a subalgebra of $\mathbf{A}$. Then $\pi_K(B) = \pi_K\iota\tau(F(A)) = \pi_K\iota(F(A)) = \pi_K(A)$, and $|\pi_L(B)| = |\pi_L\iota\tau(F(A))| = 1$. So far we have shown that $\pi_K(A) \times \{c\} \subseteq A$, for the unique element $c \in \pi_L(B)$. Since $|\pi_K(A)| > 1$ and $|\pi_L(A)| > 1$, $K \neq \emptyset$ and $L \neq \emptyset$. Thus the algebras $\pi_K(\mathbf{A})$ and $\pi_L(\mathbf{A})$ are subdirect products of less than k strongly connected tournaments, and therefore, by the induction hypothesis, they are strongly connected. Then by Lemma 5.2, $A = \pi_K(A) \times \pi_L(A)$. This contradicts that $\mathbf{A}$ is weakly indecomposable.

CLAIM 2. $\mathbf{A}$ *has a spanning triangular graph.*

Since $\mathbf{A}$ is nontrivial, there exists $i < k$ such that $|\pi_i(A)| > 1$. Then, by Lemma 5.8, the strongly connected tournament $\mathbf{T}_i$ has a spanning triangular graph $\mathbf{G}_i$. Consider the projection $\pi_i : A \to T_i$ and the directed graph $\mathbf{G} = \langle A; \pi_i^{-1}(\to^{\mathbf{G}_i})\rangle$. By Lemma 5.5, $\mathbf{G}$ is triangular, but is not necessarily compatible with $\mathbf{A}$. We argue that it is. Clearly, the set K, as defined in Claim 1, contains i, by the definiton of $\mathbf{G}$, and $|\pi_K(A)| > 1$. Thus, $x \to^{\mathbf{A}} y$ for all edges $x \to^{\mathbf{G}} y$.

Once we know that $\mathbf{A}$ has a spanning triangular graph, it has a maximal one, with respect to the number of edges, as well. It turns out that any maximal spanning triangular graph will satisfy the statement of the lemma.

Claim 3. *Let* $\mathbf{G}$ *be a maximal spanning triangular graph of* $\mathbf{A}$*. Then for all elements* $x \to^{\mathbf{A}} y \to^{\mathbf{A}} z$*, if* $x \to^{\mathbf{G}} y$ *or* $y \to^{\mathbf{G}} z$*, then* x *and* z *are comparable in* $\mathbf{A}$*.*

Assume the contrary, that there are elements $a, b, c \in A$ such that $a \to^{\mathbf{A}} b \to^{\mathbf{A}} c$, $a \to^{\mathbf{G}} b$ or $b \to^{\mathbf{G}} c$, and $ac \notin \{a, c\}$. Define $\mathbf{G}' = \langle A; \to^{\mathbf{G}} \cup \{a \to b \to c \to a\}\rangle$. We want to argue that the directed graph $\mathbf{G}'$ is triangular. Notice that $a \neq b \neq c \neq a$, because otherwise ac would be either a or c. So $\mathbf{G}'$ is the union of two triangular graphs $\mathbf{G}$ and $a \to b \to c \to a$, which have at least one common edge, $a \to b$ or $b \to c$. Now we need to show that $a \not\leftarrow^{\mathbf{G}} b \not\leftarrow^{\mathbf{G}} c \not\leftarrow^{\mathbf{G}} a$. But this holds, because $\to^{\mathbf{G}} \subseteq \to^{\mathbf{A}}$, and $a \not\leftarrow^{\mathbf{A}} b \not\leftarrow^{\mathbf{A}} c \not\leftarrow^{\mathbf{A}} a$. Thus $\mathbf{G}'$ is a triangular graph on A. Since $ac \notin \{a, c\}$, there exists $i < k$ such that $\pi_i(c) \to^{\mathbf{T}_i} \pi_i(a)$. So $\pi_i(a) \to^{\mathbf{T}_i} \pi_i(b) \to^{\mathbf{T}_i} \pi_i(c) \to^{\mathbf{T}_i} \pi_i(a)$, and therefore $i \in K$, where $K = \{\, i < k : \pi_i(x) \to^{\mathbf{T}_i} \pi_i(y)$ for all edges $x \to^{\mathbf{G}'} y \,\}$. Now by Claim 1, $x \to^{\mathbf{A}} y$ for all edges $x \to^{\mathbf{G}'} y$. In particular, $c \to^{\mathbf{A}} a$, contradicting $ac \notin \{a, c\}$.

Claim 4. *Let* $\mathbf{B}$ *be a subdirect product of at most* k *strongly connected finite tournaments. Then* $\mathbf{B}$ *is strongly connected.*

Clearly, $\mathbf{B}$ is a direct product $\prod_{i<n} \mathbf{B}_i$ of nontrivial weakly indecomposable subdirect products of at most k strongly connected tournaments. By Claim 2, each $\mathbf{B}_i$ has a spanning triangular graph, and therefore is strongly connected. Thus $\mathbf{B}$ is a direct product of strongly connected algebras, hence strongly connected. □

Triangular algebras

Definition 5.11. An algebra $\mathbf{A} \in \mathcal{T}$ is called *triangular* if $\mathbf{A}$ is isomorphic to a subdirect product of finitely many strongly connected finite tournaments, and $\mathbf{A}$ has a maximal spanning triangular graph $\mathbf{G}$ such that for all elements $x \to^{\mathbf{A}} y \to^{\mathbf{A}} z$, if $x \to^{\mathbf{G}} y$ or $y \to^{\mathbf{G}} z$, then x and z are comparable in $\mathbf{A}$.

Lemma 5.12. *Let $\mathbf{A} \in \mathcal{T}$ be a finite, strongly connected, subdirectly irreducible algebra which is a homomorphic image of a subalgebra $\mathbf{B}$ of a product of tournaments. Then there exists a triangular subalgebra $\mathbf{C} \leq \mathbf{B}$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{C}$.*

Proof. By Lemma 5.1, there exists a finite subalgebra $\mathbf{C} \leq \mathbf{B}$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{C}$. Choose $\mathbf{C}$ to be of minimal size over all such representations. Then, using Lemma 5.1 once again, $\mathbf{C}$ is a subdirect product $\mathbf{C} \leq \prod_{i<k} \mathbf{T}_i$ of finitely many strongly connected finite tournaments. We argue that $\mathbf{C}$ is weakly indecomposable. Take a partition $K \cup L$ of $\{0, 1, \ldots, k-1\}$ such that $C = \pi_K(C) \times \pi_L(C)$. Thus $\mathbf{C}$ is isomorphic to the direct product $\pi_K(\mathbf{C}) \times \pi_L(\mathbf{C})$. By Lemma 5.10, the subdirect products $\pi_K(\mathbf{C})$ and $\pi_L(\mathbf{C})$ are strongly connected. Then $\operatorname{Con} \mathbf{C} \cong \operatorname{Con} \pi_K(\mathbf{C}) \times \operatorname{Con} \pi_L(\mathbf{C})$ by Lemma 5.3. Since $\mathbf{A}$ is a homomorphic image of $\mathbf{C}$, $\mathbf{A} \cong \pi_K(\mathbf{C}) \times \pi_L(\mathbf{C})/(\alpha \times \beta)$ for some congruences $\alpha \in \operatorname{Con} \pi_K(\mathbf{C})$ and $\beta \in \operatorname{Con} \pi_L(\mathbf{C})$. But $\mathbf{A}$ is subdirectly irreducible, hence either $\alpha = 1_{\pi_K(\mathbf{C})}$ or $\beta = 1_{\pi_L(\mathbf{C})}$. Without loss of generality we can assume that $\alpha = 1_{\pi_K(\mathbf{C})}$. Thus $\mathbf{A} \cong \pi_L(\mathbf{C})/\beta$, which is another representation of $\mathbf{A}$, because $\pi_L(\mathbf{C})$ is isomorphic to a subalgebra of $\mathbf{C}$. Then, by the minimality of $|C|$, we get $|\pi_L(C)| = |C|$ and $|\pi_K(C)| = 1$. Therefore $\mathbf{C}$ is weakly indecomposable. Then, by Lemma 5.10, $\mathbf{C}$ is triangular, which concludes the proof. $\square$

Lemma 5.13. *The congruence lattice of every triangular algebra has a unique coatom.*

Proof. Let $\mathbf{A}$ be a triangular algebra with graph $\mathbf{G}$. Consider the congruence $\beta = \bigvee\{\, \alpha \in \operatorname{Con} \mathbf{A} : \alpha \neq 1_{\mathbf{A}} \,\}$. We will prove that $\beta \neq 1_{\mathbf{A}}$, that is, β is the unique maximal congruence below $1_{\mathbf{A}}$, by showing that β does not collapse any edge of $\mathbf{G}$.

Claim 1. $\operatorname{Cg}_{\mathbf{A}}(a, b) = 1_{\mathbf{A}}$ *for all edges $a \to^{\mathbf{G}} b$.*

Take elements c, d and e which form a triangle $c \to^{\mathbf{G}} d \to^{\mathbf{G}} e \to^{\mathbf{G}} c$ in $\mathbf{G}$. The unary polynomial $p(x) = xe$ maps the pair $\langle c, d\rangle$ to $\langle e, d\rangle$. Since $\mathbf{G}$ is a union of triangles connected with common edges, we can map any edge of $\mathbf{G}$ to any other by some composition of unary polynomials of the above kind. Thus, $\mathrm{Cg}_{\mathbf{A}}(a,b)$ contains all edges of $\mathbf{G}$. Since $\mathbf{G}$ is a spanning triangular graph of $\mathbf{A}$, all elements of A can be connected with $\mathbf{G}$-edges. Hence, by transitivity, $\mathrm{Cg}_{\mathbf{A}}(a,b)$ collapses all elements of A.

Claim 2. *Let $a,b,c,d \in A$ be elements such that $a \to^{\mathbf{G}} b \to^{\mathbf{G}} c \to^{\mathbf{G}} a$, and $\mathrm{Cg}_{\mathbf{A}}(c,d) \neq 1_{\mathbf{A}}$. Then $a \to^{\mathbf{G}} b \to^{\mathbf{G}} d \to^{\mathbf{G}} a$.*

Put $e = cd$. Clearly, $e \to^{\mathbf{A}} c \to^{\mathbf{G}} a$, so by the definition of triangular algebras, e and a are comparable in $\mathbf{A}$. If $a \to^{\mathbf{A}} e$, then $\langle cca, dca\rangle = \langle c,a\rangle$ and therefore $\mathrm{Cg}_{\mathbf{A}}(c,d) \supseteq \mathrm{Cg}_{\mathbf{A}}(c,a) = 1_{\mathbf{A}}$, which contradicts $\mathrm{Cg}_{\mathbf{A}}(c,d) \neq 1_{\mathbf{A}}$. So we must have $e \to^{\mathbf{A}} a$.

Now consider the edges $e \to^{\mathbf{A}} a \to^{\mathbf{G}} b$. Again, e and b must be comparable in $\mathbf{A}$. If $e \to^{\mathbf{A}} b$, then $\langle cc, dc\rangle = \langle c,e\rangle$ and $\langle cb, dcb\rangle = \langle b,e\rangle$, and therefore $\mathrm{Cg}_{\mathbf{A}}(c,d) \supseteq \mathrm{Cg}_{\mathbf{A}}(c,e) \vee \mathrm{Cg}_{\mathbf{A}}(b,e) \supseteq \mathrm{Cg}_{\mathbf{A}}(c,b) = 1_{\mathbf{A}}$. This is again a contradiction, thus $b \to^{\mathbf{A}} e$. At this point we know that $a \to^{\mathbf{G}} b \to^{\mathbf{A}} e \to^{\mathbf{A}} a$. The elements a,b and e form a triangle in $\mathbf{A}$, and this triangle has a common edge with $\mathbf{G}$. Thus, by the maximality of the spanning triangular graph $\mathbf{G}$, we get $a \to^{\mathbf{G}} b \to^{\mathbf{G}} e \to^{\mathbf{G}} a$.

Since $b \to^{\mathbf{G}} e \to^{\mathbf{A}} d$, the elements b and d are comparable in $\mathbf{A}$. If $d \to^{\mathbf{A}} b$, then $\langle cbe, dbe\rangle = \langle b,e\rangle$ and therefore $\mathrm{Cg}_{\mathbf{A}}(c,d) \supseteq \mathrm{Cg}_{\mathbf{A}}(b,e) = 1_{\mathbf{A}}$, which is a contradiction. Thus $b \to^{\mathbf{A}} d$.

Finally, consider the edges $a \to^{\mathbf{G}} b \to^{\mathbf{A}} d$. Once again, a and d must be comparable. If $a \to^{\mathbf{A}} d$, then $\langle ca, da\rangle = \langle c,a\rangle$ and therefore $\mathrm{Cg}_{\mathbf{A}}(c,d) \supseteq \mathrm{Cg}_{\mathbf{A}}(c,a) = 1_{\mathbf{A}}$, which is a contradiction. Thus $d \to^{\mathbf{A}} a$. Now we know that $a \to^{\mathbf{G}} b \to^{\mathbf{A}} d \to^{\mathbf{A}} a$, and by the maximality of $\mathbf{G}$ we conclude that $a \to^{\mathbf{G}} b \to^{\mathbf{G}} d \to^{\mathbf{G}} a$.

Claim 3. $\beta \neq 1_{\mathbf{A}}$

Take any triangle $a \to^{\mathbf{G}} b \to^{\mathbf{G}} c \to^{\mathbf{G}} a$ of $\mathbf{G}$, and assume that $\langle c,a\rangle \in \beta$. Then there exist elements $c = d_0, d_1, \dots, d_{k-1} = a$ of A such that $\mathrm{Cg}_{\mathbf{A}}(d_i, d_{i+1}) \neq 1_{\mathbf{A}}$ for all $i < k-1$. By applying the previous claim repeatedly, we get $a \to^{\mathbf{G}} b \to^{\mathbf{G}} d_i \to^{\mathbf{G}} a$ for all $i < k$. For $i = k-1$ this gives $a \to^{\mathbf{G}} a$, which clearly cannot happen in a triangular graph. Hence $\langle c, a\rangle \notin \beta$, and therefore $\beta \neq 1_{\mathbf{A}}$. □

The largest congruence of triangular algebras

Lemma 5.14. *Let* $\mathbf{A}$ *be a triangular algebra and* β *be its largest congruence below* $1_{\mathbf{A}}$*. Then* $\mathbf{A}/\beta$ *is a simple tournament. Moreover,* $a \to^{\mathbf{A}} b$ *whenever* $a/\beta \neq b/\beta$ *and* $a/\beta \to^{\mathbf{A}/\beta} b/\beta$*.*

Proof. We argue that for all pairs of elements $\langle a, b\rangle \in 1_{\mathbf{A}} \setminus \beta$, a and b are comparable. Since β is the largest congruence below $1_{\mathbf{A}}$, $\mathrm{Cg}_{\mathbf{A}}(a,b) = 1_{\mathbf{A}}$. Recall that $\mathbf{A}$ is a subdirect product $\mathbf{A} \leq \prod_{i<k} \mathbf{T}_i$ of tournaments $\mathbf{T}_i$. Define

$$K = \{\, i < k : \pi_i(a) \to^{\mathbf{T}_i} \pi_i(b) \,\},$$
$$L = \{\, i < k : \pi_i(a) \leftarrow^{\mathbf{T}_i} \pi_i(b) \,\}$$

and denote by η_K and η_L the kernels of the projections π_K and π_L of $\mathbf{A}$, respectively. We argue that $a \,\eta_K\, ab$. Clearly, $\pi_i(a) = \pi_i(a)\pi_i(b) = \pi_i(ab)$ for all $i \in K$. So, $\pi_K(a) = \pi_K(ab)$ and therefore $a \,\eta_K\, ab$. Similarly, we get $ab \,\eta_L\, b$. Now

$$1_{\mathbf{A}} = \mathrm{Cg}_{\mathbf{A}}(a,b) = \mathrm{Cg}_{\mathbf{A}}(a,ab) \vee \mathrm{Cg}_{\mathbf{A}}(ab,b) \leq \eta_K \vee \eta_L,$$

thus $1_{\mathbf{A}} = \eta_K \vee \eta_L$. This implies that either $\eta_K = 1_{\mathbf{A}}$ or $\eta_L = 1_{\mathbf{A}}$, because $\mathrm{Con}\,\mathbf{A}$ has a unique coatom. Without loss of generality, we can assume that $\eta_K = 1_{\mathbf{A}}$. In particular, $a \,\eta_K\, b$, so that $\pi_i(a) = \pi_i(b)$ for all $i \in K$. Hence $L = \{0, 1, \dots, k-1\}$, and therefore $a \leftarrow^{\mathbf{A}} b$.

Since $\beta \prec 1_{\mathbf{A}}$, $\mathbf{A}/\beta$ is simple. Take a pair $a/\beta, b/\beta$ of distinct elements of $\mathbf{A}/\beta$. Then $\langle a, b\rangle \notin \beta$, so that $ab \in \{a, b\}$ by the first part of the lemma. Thus $a/\beta \cdot b/\beta \in \{a/\beta, b/\beta\}$, and therefore $\mathbf{A}/\beta$ is a tournament.

To check the second part of the lemma, take elements $a, b \in A$ such that $a/\beta \neq b/\beta$ and $a/\beta \rightarrow^{\mathbf{A}/\beta} b/\beta$. Then $\langle a, b\rangle \notin \beta$, and therefore $ab \in \{a, b\}$. But $ab/\beta = a/\beta$, thus we must have $a \rightarrow^{\mathbf{A}} b$. $\square$

Lemma 5.15. *Let $\mathbf{A}$ be a triangular algebra and β be its largest congruence below $1_{\mathbf{A}}$. Denote the induced subalgebras of $\mathbf{A}$ on the blocks of β by $\mathbf{B}_i$, $i < k$. Then* $\operatorname{Con}\mathbf{A} \cong (\operatorname{Con}\mathbf{B}_0 \times \cdots \times \operatorname{Con}\mathbf{B}_{k-1}) \oplus 1$, *via the isomorphism which maps each congruence $\alpha \leq \beta$ to $\langle \alpha \cap B_0^2, \ldots, \alpha \cap B_{k-1}^2\rangle$.*

Proof. Because of the idempotent law, each block of β is a subuniverse of $\mathbf{A}$. Since β is the unique coatom of $\operatorname{Con}\mathbf{A}$, we need to show that there exists an isomorphism λ between the interval $[0_{\mathbf{A}}, \beta]$ of $\operatorname{Con}\mathbf{A}$ and $\operatorname{Con}\mathbf{B}_0 \times \cdots \times \operatorname{Con}\mathbf{B}_{k-1}$. Given a congruence $\alpha \leq \beta$, define

$$\lambda(\alpha) = \langle \alpha \cap B_0^2, \ldots, \alpha \cap B_{k-1}^2\rangle.$$

Clearly, $\alpha \cap B_i^2 \in \operatorname{Con}\mathbf{B}_i$ for all $i < k$, and the mapping λ is order preserving. As $\alpha \leq \beta$, λ is one-to-one. We need to check that λ is onto, as well. Given congruences $\alpha_i \in \operatorname{Con}\mathbf{B}_i$ for $i < k$, define

$$\alpha = \bigcup_{i<k} \alpha_i.$$

Clearly, $\lambda(\alpha) = \langle \alpha_0, \ldots, \alpha_{k-1}\rangle$. To check that $\alpha \in \operatorname{Con}\mathbf{A}$, take a pair $\langle x, y\rangle \in \alpha$ of elements and an element $z \in A$. We argue that $\langle xz, yz\rangle \in \alpha$, which will conclude the proof. Clearly, $x, y \in B_i$ and $z \in B_j$ for some $i, j < k$. If $i = j$, then $\langle xz, yz\rangle \in \alpha_i \subseteq \alpha$. On the other hand, if $i \neq j$, then z is comparable to both x and y, by Lemma 5.14. In fact, either $x, y \rightarrow^{\mathbf{A}} z$ or $x, y \leftarrow^{\mathbf{A}} z$, because $xz \mathrel{\beta} yz$. Hence, either $\langle xz, yz\rangle = \langle x, y\rangle$ or $\langle xz, yz\rangle = \langle z, z\rangle$, and we get $\langle xz, yz\rangle \in \alpha$. $\square$

Theorem 5.16. *Let* $\mathbf{S}$ *be a finite, strongly connected subdirectly irreducible algebra in* $\mathcal{T}$*. Then the following hold.*

(1) $\operatorname{Con}\mathbf{S}$ *has a unique coatom* γ*.*

(2) $\mathbf{S}/\gamma$ *is a simple tournament.*

Moreover, if $\gamma \neq 0_{\mathbf{S}}$*, then*

(3) γ *has exactly one nontrivial block* C*.*

(4) *For all* $x \in S \setminus C$*, either* $x \to^{\mathbf{S}} y$ *for all* $y \in C$*, or* $x \leftarrow^{\mathbf{S}} y$ *for all* $y \in C$*.*

(5) $\mathbf{S}|_C$ *is subdirectly irreducible.*

Consequently, if $\mathbf{S}$ *is not a tournament, then it has a proper subdirectly irreducible subalgebra which is again not a tournament.*

Proof. By Lemma 5.12, $\mathbf{S} \cong \mathbf{A}/\alpha$ for some triangular algebra $\mathbf{A}$ and congruence $\alpha \in \operatorname{Con}\mathbf{A}$. According to Lemma 5.13, $\operatorname{Con}\mathbf{A}$ has a unique coatom β. Clearly, $\alpha \leq \beta$. Consequently, the congruence $\gamma = \beta/\alpha$ is the unique coatom of $\operatorname{Con}\mathbf{S}$, and $\mathbf{S}/\gamma \cong \mathbf{A}/\beta$. Hence $\mathbf{S}/\gamma$ is a simple tournament by Lemma 5.14.

Now assume that $\gamma \neq 0_{\mathbf{S}}$, that is, $\alpha \neq \beta$. Let $\mathbf{B}_0, \dots, \mathbf{B}_{k-1}$ be the induced subalgebras of $\mathbf{A}$ on the blocks of β. By Lemma 5.15, $\operatorname{Con}\mathbf{A}$ is isomorphic to $(\operatorname{Con}\mathbf{B}_0 \times \cdots \times \operatorname{Con}\mathbf{B}_{k-1}) \oplus 1$ via the isomorphism which maps α to $\langle \alpha \cap B_0^2, \dots, \alpha \cap B_{k-1}^2 \rangle$. Since $\mathbf{S}$ is subdirectly irreducible, $\operatorname{Con}\mathbf{S}$ has a unique atom. Therefore, α is meet irreducible in $\operatorname{Con}\mathbf{A}$. Consequently, $\alpha \cap B_i^2 < 1_{\mathbf{B}_i}$ for at most one $i < k$. But $\alpha < \beta$, so there exists a unique $j < k$ such that $\alpha \cap B_j^2 < 1_{\mathbf{B}_j}$. Notice that the blocks of γ are the sets B_i/α. Subsequently, γ has a unique nontrivial block $C = B_j/\alpha$, which proves statement (3).

Denote by $\mathbf{C}$ the algebra $\mathbf{S}|_C$. Since $\alpha \cap B_i^2 = 1_{\mathbf{B}_i}$ for all $i \neq j$, $[\alpha, \beta] \cong [\alpha \cap B_j^2, 1_{\mathbf{B}_j}]$. On the other hand, $[\alpha \cap B_j^2, 1_{\mathbf{B}_j}] \cong [0_{\mathbf{C}}, 1_{\mathbf{C}}]$. But α is meet irreducible in $[\alpha, \beta]$, therefore $\mathbf{C}$ is subdirectly irreducible. This proves statement (5). Now notice that statement (4) follows from Lemma 5.14.

Finally, assume that $\mathbf{S}$ is not a tournament. Then $\gamma \neq 0_{\mathbf{S}}$, by (2). If $\mathbf{S}|_C$ is a tournament, then by (4), $\mathbf{S}$ is a tournament, which is a contradiction. Therefore $\mathbf{S}|_C$ is a proper subdirectly irreducible subalgebra of $\mathbf{S}$ which is again not a tournament. $\square$

Subdirectly irreducibles are tournaments

Theorem 5.17. *Every subdirectly irreducible algebra in $\mathcal{T}$ is a tournament.*

Proof. First we show that every finite subdirectly irreducible member of $\mathcal{T}$ is a tournament. To get a contradiction, take a minimal finite subdirectly irreducible algebra $\mathbf{S} \in \mathcal{T}$ which is not a tournament. By Theorem 4.11, $\mathbf{S}$ must be strongly connected. Then by Theorem 5.16, $\mathbf{S}$ has a proper subdirectly irreducible subalgebra which is again not a tournament. This contradicts the minimality of $\mathbf{S}$.

Since $\mathcal{T}$ is locally finite, every subdirectly irreducible member $\mathbf{S}$ of $\mathcal{T}$ can be embedded into an ultraproduct of finite subdirectly irreducible algebras which are homomorphic images of finite subalgebras of $\mathbf{S}$, by a result of J. B. Nation, Lemma 10.2 of [8]. The fact that a groupoid is a tournament can be expressed by the universal sentence $xy = yx \in \{x, y\}$. Since all finite subdirectly irreducible members of$\mathcal{T}$ satisfy this sentence, and ultraproducts and subalgebras preserve all universal sentences, it follows that all subdirectly irreducible members of $\mathcal{T}$ are tournaments. $\square$

CHAPTER VI

CONSEQUENCES

Immediate consequences

Theorem 5.17 answers affirmatively the conjecture of R. McKenzie posted in [13] and [14]. This result has many interesting consequences. Among others, we present a representation theorem for finite subdirectly irreducible tournaments modulo finite simple tournaments, which nicely extends Theorem 14 of [13].

Corollary 6.1. *The following hold:*

(1) *Every algebra in $\mathcal{T}$ is a subdirect product of subdirectly irreducible tournaments.*

(2) *The quasi-variety generated by tournaments is $\mathcal{T}$.*

(3) *For every quasi-equation ϕ which is satisfied in all tournaments, there exists a finite set Γ of equations of $\mathcal{T}$ such that $\Gamma \vdash \phi$.*

Proof. Every algebra in $\mathcal{T}$ is a subdirect product of subdirectly irreducible members of $\mathcal{T}$. Since all subdirectly irreducible members of $\mathcal{T}$ are tournaments, $\mathcal{T}$ is a subclass of the quasi-variety generated by tournaments. The other inclusion trivially holds.

The class $\mathcal{T}$ is a variety, so it is axiomatized by a set Σ of equations; hence $\Sigma \vdash \phi$ and, consequently, $\Gamma \vdash \phi$ for a finite subset Γ of Σ. $\square$

The blow-up representation of subdirectly irreducibles

Definition 6.2. Let $\mathbf{T}, \mathbf{S}$ be tournaments and $s \in S$. Define a tournament $\mathbf{A}$ on the disjoint union $A = T \cup S \setminus \{s\}$ by

(1) for all $a, b \in T$, $a \to^{\mathbf{A}} b$ if and only if $a \to^{\mathbf{T}} b$,

(2) for all $a, b \in S \setminus \{s\}$, $a \to^{\mathbf{A}} b$ if and only if $a \to^{\mathbf{S}} b$,

(3) for all $a \in T$ and $b \in S \setminus \{s\}$, $a \to^{\mathbf{A}} b$ if and only if $s \to^{\mathbf{S}} b$.

Clearly, $\mathbf{A}|_T = \mathbf{T}$, and $\mathbf{A}|_{\{t\} \cup S \setminus \{s\}} \cong \mathbf{S}$ for all $t \in T$. We will denote $\mathbf{A}$ by $\mathbf{T} \star s \star \mathbf{S}$.

Lemma 6.3. *Let $\mathbf{T}$ be a subdirectly irreducible tournament, $\mathbf{S}$ be a strongly connected, simple tournament, and $s \in S$. Put $\mathbf{A} = \mathbf{T} \star s \star \mathbf{S}$. Then the following hold.*

(1) *$\mathbf{A}$ is strongly connected.*

(2) *$\gamma = T^2 \cup \mathrm{id}_A$ is the largest congruence of $\mathbf{A}$ below $1_{\mathbf{A}}$.*

(3) *$\mathbf{A}|_T \cong \mathbf{T}$.*

(4) *$\mathbf{A}/\gamma \cong \mathbf{S}$ under an isomorphism sending T to s.*

(5) *$\mathrm{Con}\,\mathbf{A} \cong \mathrm{Con}\,\mathbf{T} \oplus 1$.*

(6) *$\mathbf{A}$ is subdirectly irreducible.*

Proof. Recall that $\mathbf{A}|_{\{t\} \cup S \setminus \{s\}} \cong \mathbf{S}$ for all $t \in T$. Since $\mathbf{S}$ is strongly connected, $\sim^{\mathbf{A}}$ collapses all elements of $\{t\} \cup S \setminus \{s\}$, for all $t \in T$. Hence $\sim^{\mathbf{A}}$ collapses $T \cup S \setminus \{s\}$, that is, $\mathbf{A}$ is strongly connected.

Define $\gamma = T^2 \cup \mathrm{id}_A$. Clearly, $\gamma \in \mathrm{Con}\,\mathbf{A}$. We argue that γ is the largest congruence of $\mathbf{A}$ below $1_{\mathbf{A}}$. Take a congruence $\alpha \in \mathrm{Con}\,\mathbf{A}$ such that $\alpha \not\leq \gamma$, and a pair of distinct elements $\langle a, b \rangle \in \alpha \setminus \gamma$. We can assume that $a \in S \setminus \{s\}$. If $b \in S \setminus \{s\}$, then for all $t \in T$, $\mathrm{Cg}_{\mathbf{A}}(a, b)$ collapses all elements of $\{t\} \cup S \setminus \{s\}$, because $\mathbf{S}$ is simple and $\mathbf{A}|_{\{t\} \cup S \setminus \{s\}} \cong \mathbf{S}$. This shows that $\alpha \supseteq \mathrm{Cg}_{\mathbf{A}}(a, b) = 1_{\mathbf{A}}$. On the other hand, if $b \in T$, then $\mathrm{Cg}_{\mathbf{A}}(a, b)$ collapses all elements of $\{b\} \cup S \setminus \{s\}$. In particular, it collapses $\langle a, c \rangle$ for some $c \in S \setminus \{s, a\}$. Consequently, $\alpha \supseteq \mathrm{Cg}_{\mathbf{A}}(a, b) \supseteq \mathrm{Cg}_{\mathbf{A}}(b, c) = 1_{\mathbf{A}}$, once again. The rest is obvious. $\square$

Denote by $\mathbf{2}$ and $\mathbf{3}$ the two- and three-element chains defined on the sets $\{0,1\}$ and $\{0,1,2\}$ with $0 \to 1$ and $0 \to 1 \to 2$, respectively. Let $\mathbf{T}$ be a tournament. Observe that $\mathbf{T} \star 0 \star \mathbf{2}$ is the tournament obtained from $\mathbf{T}$ by adding a new unit element (an element 1 such that $t \to 1$ for all $t \in T$). Similarly, $\mathbf{T} \star 1 \star \mathbf{2}$ is obtained from $\mathbf{T}$ by adding a new zero element (an element 0 such that $0 \to t$ for all $t \in T$), and $\mathbf{T} \star 1 \star \mathbf{3}$ is obtained from $\mathbf{T}$ by adding both a new unit and a new zero element.

Given two bounded lattices $\mathbf{K}$ and $\mathbf{L}$, the lattice $\mathbf{K} \boxplus \mathbf{L}$ is the factor lattice of $\mathbf{K} \otimes \mathbf{L}$ by the congruence with a single nontrivial block $\{1_{\mathbf{K}}, 0_{\mathbf{L}}\}$ where $1_{\mathbf{K}}$ is the largest element of $\mathbf{K}$ and $0_{\mathbf{L}}$ is the smallest element of $\mathbf{L}$. Thus, $\mathbf{2} \boxplus \mathbf{2} \cong \mathbf{3}$, for example.

Lemma 6.4. *Let $\mathbf{T}$ be a strongly connected, subdirectly irreducible tournament, and $\langle s, \mathbf{S} \rangle$ be either $\langle 0, \mathbf{2} \rangle$ or $\langle 1, \mathbf{2} \rangle$ or $\langle 1, \mathbf{3} \rangle$. Put $\mathbf{A} = \mathbf{T} \star s \star \mathbf{S}$. Then the following hold.*

(1) $\mathbf{A}$ *is not strongly connected.*

(2) $\sim$ *has a single nontrivial block T.*

(3) $\mathbf{A}|_T \cong \mathbf{T}$.

(4) $\mathbf{A}/{\sim} \cong \mathbf{S}$ *under an isomorphism sending C to s.*

(5) $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{T} \boxplus \operatorname{Con} \mathbf{S}$.

(6) $\mathbf{A}$ *is subdirectly irreducible.*

Proof. It is easy. □

Definition 6.5. Given an integer $n \geq 0$, tournaments $\mathbf{S}_0, \ldots, \mathbf{S}_n$ and elements $s_i \in S_i$, $0 < i \leq n$, we define the *blow-up composition* $\mathbf{S}_0 \star s_1 \star \mathbf{S}_1 \star \cdots \star s_n \star \mathbf{S}_n$ inductively by

$$\mathbf{S}_0 \star s_1 \star \cdots \star s_n \star \mathbf{S}_n = \begin{cases} \mathbf{S}_0 & \text{if } n = 0, \\ (\mathbf{S}_0 \star s_1 \star \cdots \star \mathbf{S}_{n-1}) \star s_n \star \mathbf{S}_n & \text{if } n > 0. \end{cases}$$

It is not hard to see that $(\mathbf{S}_0 \star s_1 \star \mathbf{S}_1) \star s_2 \star \mathbf{S}_2 \cong \mathbf{S}_0 \star s_1 \star (\mathbf{S}_1 \star s_2 \star \mathbf{S}_2)$ for all tournaments $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2$ and all elements $s_1 \in S_1$ and $s_2 \in S_2$. Thus the arrangement of parentheses according to which the blow-up composition is computed does not matter.

Theorem 6.6. *Every finite subdirectly irreducible tournament can be uniquely represented as $\mathbf{S}_0 \star s_1 \star \mathbf{S}_1 \star \cdots \star s_n \star \mathbf{S}_n$, where:*

(1) *Each $\mathbf{S}_i$ is either a two- or three-element chain or a finite, strongly connected simple tournament.*

(2) *There is no index $i < n$ such that both $\mathbf{S}_i$ and $\mathbf{S}_{i+1}$ are chains.*

(3) *$s_i \in S_i$ for all $0 < i \leq n$.*

(4) *$\mathbf{S}_0$ is not a three-element chain. If $\mathbf{S}_i$ is a three-element chain, then s_i is the middle element of $\mathbf{S}_i$.*

Moreover, $\mathbf{S}_0 \star s_1 \star \cdots \star s_n \star \mathbf{S}_n$ is a finite subdirectly irreducible tournament for all choices of $\mathbf{S}_0, \ldots, \mathbf{S}_n$ and $s_1, \ldots, s_n$ for which $(1) - (4)$ hold.

Proof. Throughout this proof we consider only finite algebras. Let $\mathbf{A}$ be a subdirectly irreducible tournament.

Claim 1. *If $\mathbf{A}$ is simple, then either $\mathbf{A} \cong \mathbf{2}$, or $\mathbf{A}$ is strongly connected. Consequently, $\mathbf{A}$ can be uniquely represented as the blow-up composition $\mathbf{A}$ of length 1.*

Consider the congruence $\sim$ of $\mathbf{A}$ from Lemma 4.2. If $\sim\, = 0_{\mathbf{A}}$, then $\mathbf{A}$ is a simple semilattice, therefore $\mathbf{A} \cong \mathbf{2}$. On the other hand, if $\sim\, = 1_{\mathbf{A}}$, then $\mathbf{A}$ is strongly connected. Notice that this representation is unique, because all blow-up compositions of length more than one are not simple by Lemmas 6.3 and 6.4.

Claim 2. *If $\mathbf{A}$ is not simple and is strongly connected, then $\mathbf{A}$ can be uniquely represented as $\mathbf{T} \star s \star \mathbf{S}$ where $\mathbf{T}$ is a subdirectly irreducible tournament, $\mathbf{S}$ is a strongly connected simple tournament, and $s \in S$.*

By Theorem 5.16, $\mathbf{A}$ has a largest congruence γ, $\mathbf{A}/\gamma$ is a strongly connected simple tournament, γ has a unique nontrivial block C, $\mathbf{A}|_C$ is subdirectly irreducible, and $\mathbf{A} \cong \mathbf{A}|_C \star C \star \mathbf{A}/\gamma$. On the other hand, for every representation $\mathbf{A} \cong \mathbf{T} \star s \star \mathbf{S}$, $\mathbf{A}|_C \cong \mathbf{T}$ and $\mathbf{A}/\gamma \cong \mathbf{S}$ under an isomorphism sending C to s, by Lemma 6.3.

Claim 3. *If $\mathbf{A}$ is not simple and not strongly connected, then $\mathbf{A}$ can be uniquely represented as either $\mathbf{T} \star 0 \star \mathbf{2}$ or $\mathbf{T} \star 1 \star \mathbf{2}$ or $\mathbf{T} \star 1 \star \mathbf{3}$, where $\mathbf{T}$ is a strongly connected subdirectly irreducible tournament.*

Consider the congruence $\sim$ of $\mathbf{A}$. Since $\mathbf{A}$ is a tournament, $\mathbf{A}/{\sim}$ is a chain. If ${\sim} = 0_{\mathbf{A}}$, then $\mathbf{A}$ is a subdirectly irreducible chain, that is, $\mathbf{A} \cong \mathbf{2}$. This contradicts the assumption that $\mathbf{A}$ is not simple, therefore $\sim$ has at least one nontrivial block. Clearly, for every block B of $\sim$, $B^2 \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$, and the meet of any pair of such congruences is $0_{\mathbf{A}}$ (if they are distinct congruences). This shows that $\sim$ has exactly one nontrivial block C, and as a consequence, $\mathbf{A} \cong \mathbf{A}|_C \star C \star \mathbf{A}/{\sim}$.

If the chain $\mathbf{A}/{\sim}$ has more than 3 elements, then there exists a pair of elements $a, b \in A \setminus C$ such that b covers a. Then $\mathrm{Cg}_{\mathbf{A}}(a,b) = \{a,b\}^2 \cup \mathrm{id}_A$, and ${\sim} \cap \mathrm{Cg}_{\mathbf{A}}(a,b) = 0_{\mathbf{A}}$, contradicting that $\mathbf{A}$ is subdirectly irreducible. Consequently, $\mathbf{A}/{\sim}$ is either a two- or three-element chain. If $\mathbf{A}/{\sim} \cong \mathbf{3}$ and C is not the middle element, then we can choose a cover $a, b \in A \setminus C$ yielding a contradiction, once again. This shows that $\langle C, \mathbf{A}/{\sim} \rangle$ is isomorphic to either $\langle 0, \mathbf{2} \rangle$ or $\langle 1, \mathbf{2} \rangle$ or $\langle 1, \mathbf{3} \rangle$.

From Lemma 6.4 and from the above description it follows that this representation is unique.

Now the first statement of the theorem follows from the previous claims by induction on the size of $\mathbf{A}$. The second part follows from Lemmas 6.3 and 6.4. $\square$

Definition 6.7. Given a finite subdirectly irreducible tournament $\mathbf{A}$, the *blow-up representation of* $\mathbf{A}$ is the unique blow-up composition $\mathbf{S}_0 \star s_1 \star \cdots \star s_n \star \mathbf{S}_n$ satisfying (1) – (4) of Theorem 6.6.

The hereditarily zeroless companion

Definition 6.8. Let $\mathbf{A}$ be a finite subdirectly irreducible tournament with blow-up representation $\mathbf{S}_0 \star s_1 \star \cdots \star s_n \star \mathbf{S}_n$. We say that $\mathbf{A}$ is *hereditarily zeroless* if the subsequences $1 \star \mathbf{2}$ and $1 \star \mathbf{3}$ do not occur in $\mathbf{S}_0 \star s_1 \star \cdots \star s_n \star \mathbf{S}_n$. The *hereditarily zeroless companion* of $\mathbf{A}$ is the finite subdirectly irreducible tournament whose blow-up representation is obtained from $\mathbf{S}_0 \star s_1 \star \cdots \star s_n \star \mathbf{S}_n$ by removing all occurrences of $1 \star \mathbf{2}$, and by replacing all occurrences of $1 \star \mathbf{3}$ with $0 \star \mathbf{2}$.

Lemma 6.9. *Let $\mathbf{A}$ be a finite subdirectly irreducible tournament and $\mathbf{B}$ be its hereditarily zeroless companion. Then $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{B})$.*

Proof. The following two claims prove the lemma by induction on the length of the blow-up representation of $\mathbf{A}$.

Claim 1. *Let $\mathbf{C}, \mathbf{D}$ and $\mathbf{S}$ be tournaments, and $s \in S$. If $\mathcal{V}(\mathbf{C}) = \mathcal{V}(\mathbf{D})$, then $\mathcal{V}(\mathbf{C} \star s \star \mathbf{S}) = \mathcal{V}(\mathbf{D} \star s \star \mathbf{S})$.*

Since $\mathbf{C} \in \mathcal{V}(\mathbf{D})$, there exist a cardinal κ, a subalgebra $\mathbf{E}$ of $\mathbf{D}^\kappa$, and a congruence ϑ of $\mathbf{E}$ such that $\mathbf{E}/\vartheta \cong \mathbf{C}$. Clearly, $\mathbf{E} \star s \star \mathbf{S}$ is a subalgebra of $(\mathbf{D} \star s \star \mathbf{S})^k$ if we identify the elements $t \in S \setminus \{s\}$ of $\mathbf{E} \star s \star \mathbf{S}$ with the constant κ-tuples $\langle t, t, \dots \rangle$ of $(\mathbf{D} \star s \star \mathbf{S})^\kappa$. On the other hand, $\vartheta' = \vartheta \cup \mathrm{id}_{\mathbf{E} \star s \star \mathbf{S}}$ is a congruence of $\mathbf{E} \star s \star \mathbf{S}$, and $\mathbf{E} \star s \star \mathbf{S}/\vartheta' \cong \mathbf{C} \star s \star \mathbf{S}$. This proves that $\mathcal{V}(\mathbf{C} \star s \star \mathbf{S}) \subseteq \mathcal{V}(\mathbf{D} \star s \star \mathbf{S})$. The inclusion in the other direction holds by symmetry.

Claim 2. *Let $\mathbf{C}$ be a nontrivial tournament. Then $\mathcal{V}(\mathbf{C} \star 1 \star \mathbf{2}) = \mathcal{V}(\mathbf{C})$ and $\mathcal{V}(\mathbf{C} \star 1 \star \mathbf{3}) = \mathcal{V}(\mathbf{C} \star 0 \star \mathbf{2})$.*

Recall that $\mathbf{C} \star 1 \star \mathbf{2}$ is the tournament obtained from $\mathbf{C}$ by adding a new zero element. Clearly, $\mathbf{C}$ is a subalgebra of $\mathbf{C} \star 1 \star \mathbf{2}$. On the other hand, $\mathbf{C} \star 1 \star \mathbf{2}$

is isomorphic to $(\mathbf{C} \times \mathbf{2})/\vartheta$ where $\vartheta = (C \times \{0\})^2 \cup \mathrm{id}_{\mathbf{C}\times\mathbf{2}}$. Since $\mathbf{C}$ is not trivial, $\mathbf{2} \in \mathcal{V}(\mathbf{C})$, and therefore $\mathbf{C} \star 1 \star \mathbf{2} \in \mathcal{V}(\mathbf{C})$.

The second equality follows from the first, because $\mathbf{C}\star 1\star\mathbf{3} \cong (\mathbf{C}\star 0\star\mathbf{2})\star 1\star\mathbf{2}$. $\square$

The following theorem, its proof and many of its consequences were inspired by ideas of R. McKenzie in [15].

Theorem 6.10. *Let $\mathbf{A}$ be a finite subdirectly irreducible tournament in the variety generated by a collection $\mathcal{K}$ of tournaments. Then the hereditarily zeroless companion of $\mathbf{A}$ is a subalgebra of a member of $\mathcal{K}$.*

Proof. Without loss of generality we can assume that $\mathbf{A}$ is hereditarily zeroless, by Lemma 6.9. Write $\mathbf{A}$ as a quotient of a finite subalgebra $\mathbf{B}$ of a product of finitely many tournaments in $\mathcal{K}$. We argue that $\mathbf{A}$ can be embedded into $\mathbf{B}$ by induction on the size of $\mathbf{A}$. This will imply that $\mathbf{A}$ can be embedded into some member of $\mathcal{K}$, because $\mathbf{A}$ is subdirectly irreducible.

Claim 1. *If $\mathbf{A}$ is strongly connected, then $\mathbf{A}$ can be embedded into $\mathbf{B}$.*

Assume that $\mathbf{A}$ is strongly connected. By Lemma 5.12, there exists a triangular subalgebra $\mathbf{C} \leq \mathbf{B}$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{C}$. Let $\alpha \in \mathrm{Con}\,\mathbf{C}$ for which $\mathbf{A} \cong \mathbf{C}/\alpha$, and β be the largest congruence of $\mathbf{C}$ below $1_{\mathbf{C}}$, which exists by Lemma 5.13. If $\alpha = 1_{\mathbf{A}}$, then $\mathbf{A}$ is trivial and $\mathbf{A} \leq \mathbf{B}$. So assume that $\alpha \leq \beta$. Denote the induced subalgebras of $\mathbf{C}$ on the blocks of β by $\mathbf{B}_0, \dots, \mathbf{B}_{t-1}$. By Lemma 5.15, $[0_{\mathbf{A}}, \beta] \cong \mathrm{Con}\,\mathbf{B}_0 \times \cdots \times \mathrm{Con}\,\mathbf{B}_{t-1}$ via the isomorphism which maps α to $\langle \alpha \cap B_0^2, \dots, \alpha \cap B_{t-1}^2 \rangle$. Since $\mathbf{C}/\alpha$ is subdirectly irreducible, α is meet irreducible and thus $\alpha \cap B_i^2 < 1_{\mathbf{B}_j}$ for at most one $i < t$. First consider the case when $\alpha \cap B_i^2 = 1_{\mathbf{B}_i}$ for all $i < t$, that is, $\alpha = \beta$. Take representative elements $b_i \in B_i$ for all $i < t$. Now by Lemma 5.14, $\mathbf{C}/\beta$ is a simple tournament which is isomorphic to the induced subalgebra of $\mathbf{C}$ on $\{b_0, \dots, b_{t-1}\}$.

Now assume that $\alpha < \beta$, that is, $\alpha \cap B_j^2 < 1_{\mathbf{B}_j}$ for a unique index $j < t$. Then $\mathbf{A}' = \mathbf{B}_j/(\alpha \cap B_j^2)$ is a subdirectly irreducible algebra, $|A'| < |A|$, and $\mathbf{B}_j \leq \mathbf{B}$. Using the induction hypothesis, $\mathbf{A}'$ can be embedded into a subalgebra $\mathbf{B}' \leq \mathbf{B}_j$. Consider the induced subalgebra $\mathbf{C}'$ of $\mathbf{C}$ on the set $B' \cup \{b_0, \ldots, b_{j-1}, b_{j+1}, \ldots, b_{t-1}\}$. By Lemma 5.14, $\mathbf{C}'$ is isomorphic to $\mathbf{C}/\alpha$, which concludes the proof of the claim.

Claim 2. *If* $\mathbf{A}$ *is not strongly connected, then* $\mathbf{A}$ *can be embedded into* $\mathbf{B}$*, once again.*

If $\mathbf{A} \cong \mathbf{2}$, then $\mathbf{A}$ can be clearly embedded into $\mathbf{B}$. So assume that $\mathbf{A}$ has more than 2 elements. Consider the blow-up representation of $\mathbf{A}$. Since $\mathbf{A}$ is not strongly connected, $\mathbf{A} \cong \mathbf{T} \star s \star \mathbf{S}$ for some strongly connected subdirectly irreducible tournament $\mathbf{T}$, a two- or three-element chain $\mathbf{S}$ and $s \in S$. But $\mathbf{A}$ is hereditarily zeroless, so we must have $\mathbf{S} \cong \mathbf{2}$ and $s = 0$. Thus $\mathbf{A}$ is obtained from $\mathbf{T}$ by adding a new unit element 1, and $A = T \cup \{1\}$.

Let φ be a homomorphism of $\mathbf{B}$ onto $\mathbf{A}$. Take an element $u \in \varphi^{-1}(1)$. For each $t \in T$ we can choose a representative $b \in \varphi^{-1}(t)$ such that $b \to^{\mathbf{B}} u$, because $t \to^{\mathbf{A}} 1$. Denote by $\mathbf{C}$ the subalgebra of $\mathbf{B}$ generated by the set of representatives of the elements of T. Clearly, φ maps $\mathbf{C}$ onto $\mathbf{T}$. Using the induction hypothesis for $\mathbf{T}$, there exists a subalgebra $\mathbf{D} \leq \mathbf{C}$ such that $\mathbf{T} \cong \mathbf{D}$. On the other hand, $c \to^{\mathbf{B}} u$ for all $c \in C$, by Lemma 3.4, because this holds for all generators of $\mathbf{C}$. In particular, $d \to^{\mathbf{B}} u$ for all $d \in D$. This shows that $D \cup \{u\}$ is a subuniverse of $\mathbf{B}$, and the induced subalgebra is isomorphic to $\mathbf{A}$. $\square$

Finitely generated subvarieties of $\mathcal{T}$

Corollary 6.11. *Every finitely generated subvariety of* $\mathcal{T}$ *has a finite residual bound.*

Proof. Take a finitely generated subvariety $\mathcal{V}$ of $\mathcal{T}$. Every finitely generated variety is generated by a single finite algebra, so $\mathcal{V} = \mathcal{V}(\mathbf{A})$ for some finite algebra $\mathbf{A} \in \mathcal{T}$.

The algebra $\mathbf{A}$ is a subdirect product of subdirectly irreducible finite tournaments $\mathbf{T}_0, \ldots, \mathbf{T}_{k-1}$. Clearly, $\mathbf{T}_i \in \mathcal{V}(\mathbf{A})$ for all $i < k$. Thus $\mathcal{V} = \mathcal{V}(\mathbf{T}_0, \ldots, \mathbf{T}_{k-1})$. By Theorem 6.10, every hereditarily zeroless companion of a finite subdirectly irreducible member of $\mathcal{V}$ can be embedded into one of the tournaments $\mathbf{T}_0, \ldots, \mathbf{T}_{k-1}$. Together with Lemma 6.9 this implies that $\mathcal{V}$ has only finitely many finite subdirectly irreducibles, up to isomorphism. Also, $\mathcal{V}$ has no infinite subdirectly irreducible members, by Theorem §V 3.8 of [1]. Hence $\mathcal{V}$ has a finite residual bound. $\square$

Corollary 6.12. *Every finitely generated subvariety of $\mathcal{T}$ is finitely based. Consequently, every finite tournament is finitely based.*

Proof. The variety $\mathcal{T}$ is locally finite by Corollary 3.2, and congruence meet-semidistributive by Theorem 3.7. Every finitely generated subvariety $\mathcal{V}$ of $\mathcal{T}$ has a finite residual bound by Corollary 6.11. Then, by the main theorem of [21], $\mathcal{V}$ is finitely based. $\square$

The lattice of subvarieties of $\mathcal{T}$

Let $\mathcal{Z}$ be a representative set of hereditarily zeroless, finite subdirectly irreducible tournaments, that is, a set such that each hereditarily zeroless finite subdirectly irreducible tournament is isomorphic to exactly one member of $\mathcal{Z}$. Clearly, the relation $\leq$, defined by $\mathbf{A} \leq \mathbf{B}$ if and only if $\mathbf{A}$ can be embedded into $\mathbf{B}$, is a partial order on $\mathcal{Z}$.

Corollary 6.13. *The lattice of subvarieties of $\mathcal{T}$ is isomorphic to the lattice of downsets of $\mathcal{Z}$.*

Proof. Let $\mathcal{V}$ be a subvariety of $\mathcal{T}$. Clearly, $\mathcal{V} \cap \mathcal{Z}$ is a downset of $\mathcal{Z}$. On the other hand, since $\mathcal{V}$ is locally finite, $\mathcal{V}$ is generated by its finite subdirectly irreducible members. Then by Lemma 6.9, $\mathcal{V}$ is generated by $\mathcal{V} \cap \mathcal{Z}$. This shows that $\varphi : \mathcal{V} \mapsto \mathcal{V} \cap \mathcal{Z}$ is a one-to-one mapping of the set of subvarieties of $\mathcal{T}$ to the set of downsets of $\mathcal{Z}$. It is not hard to see that φ is onto, as well, by Theorem 6.10. Finally, φ preserves intersection, which proves that φ is a lattice isomorphism. $\square$

Corollary 6.14. *The lattice of subvarieties of $\mathcal{T}$ is completely distributive. Every completely join irreducible subvariety of $\mathcal{T}$ is generated by a unique (up to isomorphism), hereditarily zeroless, finite subdirectly irreducible tournament.*

Proof. We use Corollary 6.13. Clearly, the lattice of downsets of any partially ordered set is completely distributive, and the completely join irreducible elements correspond to the principal downsets. The downset generated by a hereditarily zeroless, finite subdirectly irreducible tournament $\mathbf{A} \in \mathcal{Z}$ corresponds to the completely join irreducible subvariety of $\mathcal{T}$ generated by $\mathbf{A}$. $\square$

Corollary 6.15. *The lattice $\mathbf{2}^\omega$ can be embedded into the lattice of subvarieties of $\mathcal{T}$. Consequently, $\mathcal{T}$ has uncountably many subvarieties.*

Proof. By Theorem 3.13, there exists an infinite sequence of finite simple tournaments $\mathbf{A}_n$ ($n \geq 8$) such that no one is isomorphic to a subalgebra of some other one. Since each $\mathbf{A}_n$ is simple and has no zero and no unit element, $\mathbf{A}_n$ is hereditarily zeroless. Then the varieties $\mathcal{V}(S)$ generated by all subsets $S \subseteq \{\mathbf{A}_8, \mathbf{A}_9, \dots\}$ are pairwise distinct by Theorem 6.10. $\square$

Corollary 6.16. *Let $\mathbf{A}$ be a hereditarily zeroless, finite subdirectly irreducible tournament. Then $\mathbf{A}$ is a splitting algebra in $\mathcal{T}$. Consequently, there exists an equation ε such that for all tournaments $\mathbf{T}$, $\mathbf{A}$ can be embedded into $\mathbf{T}$ if and only if ε fails in $\mathbf{T}$.*

Proof. The variety $\mathcal{V} = \mathcal{V}(\mathbf{A})$ is a completely join irreducible subvariety of $\mathcal{T}$. Clearly, $\mathcal{V} \cap \mathcal{Z}$ is the principal downset of $\mathcal{Z}$ generated by $\mathbf{A}$. Let $\mathcal{W}$ be the variety generated by the downset $\{\, \mathbf{B} \in \mathcal{Z} : \mathbf{A} \not\leq \mathbf{B} \,\}$. Clearly, $\mathcal{W}$ is a completely meet irreducible subvariety of $\mathcal{T}$, and the pair $\mathcal{V}, \mathcal{W}$ splits the lattice of subvarieties of $\mathcal{T}$. Now there exists a set Σ of equations such that $\mathcal{W}$ is the class of all algebras in $\mathcal{T}$ satisfying all equations in Σ. Since $\mathcal{W}$ is completely meet irreducible, there exits an equation $\varepsilon \in \Sigma$ such that $\mathcal{W}$ is the class of all algebras in $\mathcal{T}$ satisfying ε. Now for all algebras $\mathbf{B} \in \mathcal{T}$, either $\mathbf{B} \in \mathcal{V}$ or else $\mathbf{B}$ satisfies the equation ε. □

Maximal spanning triangular subgraphs

To fully understand the lattice of subvarieties of $\mathcal{T}$, we need to understand the poset $\mathcal{Z}$ of hereditarily zeroless, finite subdirectly irreducible tournaments, by Corollary 6.14. As the first step in this direction we need to know more about the structure of simple tournaments. The following results are due to R. McKenzie, and are reproduced from [16].

Theorem 6.17. *A tournament has at most one maximal spanning triangular subgraph.*

Proof. Clearly, two maximal triangular subgraphs are either equal, or have no edge in common. So to prove this theorem, we assume that $\mathbf{T}$ is a tournament with maximal spanning triangular graphs $\mathbf{G}_0, \mathbf{G}_1$ which have no edges in common. "Spanning" means that every element of $\mathbf{T}$ is incident with an edge of $\mathbf{G}_0$ and with an edge of $\mathbf{G}_1$. We work toward a contradiction.

First, we work toward demonstrating that there are no two vertex-disjoint triangles $\mathbf{T}_0, \mathbf{T}_1$ with $\mathbf{T}_i \subseteq \mathbf{G}_i$. Suppose, to the contrary that $\mathbf{T}_0, \mathbf{T}_1$ are such. By a "homogeneous vertex" of the graph $\mathbf{T}_0 \cup \mathbf{T}_1$ we mean a vertex $x \in T_i$, where $\{i, j\} = \{0, 1\}$, such that either $x \to^{\mathbf{T}} T_j$ or else $T_j \to^{\mathbf{T}} x$.

Claim 1. *The graph $\mathbf{T}_0 \cup \mathbf{T}_1$ has a homogenous vertex.*

To see this, let $\mathbf{T}_i$ consist of the vertices and edges $a_i \to^{\mathbf{G}_i} b_i \to^{\mathbf{G}_i} c_i \to^{\mathbf{G}_i} a_i$. We assume that there are no homogeneous vertices and work to a contradiction. Clearly, we can assume, then, that $b_1 \to^{\mathbf{T}} a_0 \to^{\mathbf{T}} a_1 \to^{\mathbf{G}_1} b_1$. Thus by maximality,

$b_1 \to^{\mathbf{G}_1} a_0 \to^{\mathbf{G}_1} a_1$. Then if $b_1 \to^{\mathbf{G}_1} a_0 \to^{\mathbf{G}_0} b_0 \to^{\mathbf{T}} b_1$, it follows that this triangle belongs both to $\mathbf{G}_0$ and $\mathbf{G}_1$, a contradiction. Hence it must be that $b_1 \to^{\mathbf{T}} b_0$. Then $b_1 \to a_0, b_1 \to b_0$ imply that $c_0 \to b_1$, else b_1 is homogeneous. Now we have the triangle $b_1 \to^{\mathbf{T}} b_0 \to^{\mathbf{G}_0} c_0 \to^{\mathbf{T}} b_1$ putting all these edges in $\mathbf{G}_0$; and by the same argument as above, we must have $c_0 \to^{\mathbf{T}} c_1$. Since $c_0 \to b_1, c_0 \to c_1$, then $a_1 \to c_0$ because c_0 is not homogeneous. The triangle $c_0 \to^{\mathbf{T}} c_1 \to^{\mathbf{G}_1} a_1 \to^{\mathbf{T}} c_0$ puts all these edges in $\mathbf{G}_1$, and as above, forces $a_1 \to a_0$. Note now that we began with the assumption that $a_0 \to^{\mathbf{T}} a_1$. This contradiction proves the claim.

Claim 2. *Suppose that $\{i, j\} = \{0, 1\}$ and that $x \to^{\mathbf{T}_i} y$. Then $T_j \to^{\mathbf{T}} x$ implies y is homogeneous; while $y \to^{\mathbf{T}} T_j$ implies x is homogeneous.*

Indeed, suppose that $T_j \to x$ but that y is not homogeneous. Then we have some $u \to^{\mathbf{T}_j} v$ with $v \to^{\mathbf{T}} y \to^{\mathbf{T}} u$, forming a triangle. We also have the triangle $u \to^{\mathbf{T}} x \to^{\mathbf{G}_i} y \to^{\mathbf{T}} u$. The first triangle must have all its edges in $\mathbf{G}_j$, the second has all its edges in $\mathbf{G}_i$, by maximality. Then the edge $y \to u$ belongs to both $\mathbf{G}_0$ and $\mathbf{G}_1$, a contradiction. The proof that $y \to^{\mathbf{T}} T_j$ implies x is homogeneous, is entirely analogous.

Claim 3. *For the vertex disjoint triangles $\mathbf{T}_i \subseteq \mathbf{G}_i$, we must have either $T_0 \to^{\mathbf{T}} T_1$, $T_1 \to^{\mathbf{T}} T_0$, or for some $\{i, j\} = \{0, 1\}$, there is $x \to^{\mathbf{T}_i} y$ such that*

$$T_j \to^{\mathbf{T}} x \to^{\mathbf{T}_i} y \to^{\mathbf{T}} T_j\,.$$

Claim 3 follows trivially from Claims 1 and 2.

Continuing our proof that the vertex disjoint triangles $\mathbf{T}_i \subseteq \mathbf{G}_i$ cannot exist, assume now that the second alternative in Claim 3 holds, say

$$T_1 \to^{\mathbf{T}} x \to^{\mathbf{T}_0} y \to^{\mathbf{T}} T_1\,.$$

The vertex x belongs to some triangle in $\mathbf{G}_1$ (since $\mathbf{G}_1$ is spanning). In fact, by the definition of triangular graph, x belongs to a triangle $\mathbf{S}$ in $\mathbf{G}_1$ such that there is a sequence of triangles $\mathbf{T}_1 = \mathbf{S}_1, \ldots, \mathbf{S}_n = \mathbf{S}$ in $\mathbf{G}_1$ such that $\mathbf{S}_i$, $\mathbf{S}_{i+1}$ have an edge in common, for each i. Now it is easy to show, inductively, that $\{x, y\}$ is disjoint from the vertex set of each $\mathbf{S}_i$ and in fact, $y \to^{\mathbf{T}} S_i \to^{\mathbf{T}} x$ for all i. Since x belongs to the vertex set of $\mathbf{S}_n$, this gives $y \to^{\mathbf{T}} x$, which is a contradiction. Thus the second alternative in Claim 3 can never hold for a pair of vertex disjoint triangles, one in $\mathbf{G}_0$, the other in $\mathbf{G}_1$.

So now assume that $\mathbf{T}_i \subseteq \mathbf{G}_i$ are vertex disjoint, and say $T_0 \to^{\mathbf{T}} T_1$. Again, where $x \to^{\mathbf{T}_0} y \to^{\mathbf{T}_0} z \to^{\mathbf{T}_0} x$, there is a sequence of triangles $\mathbf{T}_1 = \mathbf{S}_1, \ldots, \mathbf{S}_n$ in $\mathbf{G}_1$ such that x is a vertex of $\mathbf{S}_n$ and $\mathbf{S}_i, \mathbf{S}_{i+1}$ have an edge in common, for each i. Suppose we have shown inductively, for a certain $i < n$, that $\mathbf{S}_i$ is vertex-disjoint from $\mathbf{T}_0$ and that $T_0 \to S_i$. If $\mathbf{S}_{i+1}$ is vertex-disjoint from $\mathbf{T}_0$, then since $\mathbf{S}_{i+1}$ has a vertex u with $T_0 \to u$, it follows that $T_0 \to S_{i+1}$ (by the fact that the second alternative in Claim 3 cannot hold, as we've shown above). On the other hand, if the vertex in $S_{i+1} \setminus S_i$ belongs to T_0, then for one of the two vertices $u \in S_i \cap S_{i+1}$, we do not have $T_0 \to u$, contradicting our assumption that $T_0 \to S_i$. The conclusion is that $T_0 \to S_i$ and $S_i \cap T_0 = \emptyset$ is forced for all i. Since x is a vertex of $\mathbf{S}_n$, this of course is the final contradiction.

So we have shown that every pair of triangles, consisting of one included in $\mathbf{G}_0$ and one included in $\mathbf{G}_1$, must have a vertex in common. Obviously, since $\mathbf{G}_0$ and $\mathbf{G}_1$ have no edge in common, and each contains a triangle, then $|T| \geq 4$. Thus by the definition of triangular graph, there must exist two distinct triangles $\mathbf{T}_1, \mathbf{T}_2$ contained in $\mathbf{G}_1$ which have an edge in common. Say $\mathbf{T}_i$ consists of $a \to b \to c_i$, $i \in \{1, 2\}$. There is a triangle $\mathbf{T}_0$ in $\mathbf{G}_0$ which includes the vertex c_1. Since $\mathbf{T}_0$ can have only the one vertex in common with $\mathbf{T}_1$, then it has neither a nor b. Since $\mathbf{T}_0$ must have a vertex from $\mathbf{T}_2$, then $\mathbf{T}_0$ consists of, say $c_1 \to c_2 \to c_3 \to c_1$. If $c_3 \to b$ then the triangle $b \to c_2 \to c_3 \to b$ has an edge in commong with $\mathbf{T}_0$ and an edge in common with $\mathbf{T}_2$, so that it must be included in $\mathbf{G}_0 \cap \mathbf{G}_1$, a contradiction. Thus $b \to^{\mathbf{T}} c_3$.

If $a \to c_3$, then the triangle $c_3 \to c_1 \to a \to c_3$ has an edge in common with $\mathbf{T}_0$ and an edge in common with $\mathbf{T}_1$, so that we have the same contradiction.

Thus, finally, we have the triangle $a \to b \to c_3 \to a$, and it obviously is included in $\mathbf{G}_1$—call this triangle $\mathbf{T}_3$.

Let $\mathbf{S} \subseteq \mathbf{G}_0$ be any triangle of $\mathbf{G}_0$ having an edge in common with $\mathbf{T}_0$. Since $\mathbf{S}$ contains some c_i and must intersect $\mathbf{T}_i$ in exactly one vertex, it follows that a, b are not vertices of $\mathbf{S}$. Since $\mathbf{S}$ cannot be vertex disjoint from any of $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$, then $\mathbf{S}$ has exactly the same three vertices as $\mathbf{T}_0$; i.e., $\mathbf{S} = \mathbf{T}_0$. Now, a quick glance at the definition of triangular graph leads us to the conclusion that the universe of the tournament $\mathbf{T}$ is just the set $\{c_1, c_2, c_3\}$. This is the final contradiction that proves the theorem. $\square$

Theorem 6.18. *Suppose that the tournament* $\mathbf{T}$ *has maximal spanning triangular subgraph* $\mathbf{G}$*. Then* $\mathbf{T}$ *is simple iff every triangle in* $\mathbf{T}$ *has its edges in* $\mathbf{G}$ *and there is no two-element subset* $\{x, y\} = E$ *such that for all* $z \in T$*,* $E \to z$ *or* $z \to E$*.* $\square$

Theorem 6.19. *Let* $\mathbf{T}$ *be a finite simple tournament of more than two elements, and* $\mathbf{G}$ *be its unique maximal spanning triangular subgraph. The relation*

$$\{(x, y) : x \to^{\mathbf{T}} y \text{ and } x \not\to^{\mathbf{G}} y \text{ and } y \not\to^{\mathbf{G}} x\}$$

is a partial ordering of T *and* $\mathbf{G}$ *consists of an assignment of directions on the (symmetric) incomparability graph of this ordering.* $\square$

Theorem 6.20. *For* $n \geq 1$*, there is an* n*-element triangular tournament whose associated partial ordering (as above) is discrete iff* $n \notin \{1, 2, 4\}$*.* $\square$

Theorem 6.21. *Suppose that* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are two tournaments on the same set finite*

T of vertices and $\mathbf{G}$ is a triangular graph on T which is a maximal spanning triangular subgraph for both $\mathbf{T}_1$ and $\mathbf{T}_2$. If $\mathbf{T}_1$ is simple, then $\mathbf{T}_1 = \mathbf{T}_2$.

Proof. Let $\leq_1$ and $\leq_2$ be the associated partial orders. Thus for all $x \neq y$ in T holds precisely one of the following: $x <_1 y$, $y <_1 x$, $x \to^{\mathbf{G}} y$, $y \to^{\mathbf{G}} x$; moreover $x \to^{\mathbf{T}_1} y$ iff $x <_1 y$ or $x \to^{\mathbf{G}} y$.

Now suppose that $x \prec_1 y$, i.e., y covers x in $\leq_1$. Since $\mathbf{T}_1$ is simple, there is $z \in T \setminus \{x, y\}$ such that $x \to^{\mathbf{T}} z \to^{\mathbf{T}} y$. Since $x \prec_1 y$, then either $x \to^{\mathbf{G}} z$ or $z \to^{\mathbf{G}} y$. There are three cases to consider.

Case 1: $x \to^{\mathbf{G}} z \to^{\mathbf{G}} y$. In this case, if $y \to^{\mathbf{T}_2} x$, then the $\mathbf{T}_2$ triangle consisting of x, z, y, together with $x \to^{\mathbf{G}} z$, forces $y \to^{\mathbf{G}} x$, contradicting that $x \to^{\mathbf{T}_1} y$. Hence we conclude that in this case, $x \to^{\mathbf{T}_2} y$.

Case 2: $x \to^{\mathbf{G}} z <_1 y$. Again, suppose that $y \to^{\mathbf{T}_2} x$. Since it is not the case that $y \to^{\mathbf{G}} x$, then y, x, z does not constitute a $\mathbf{T}_2$-triangle, implying that we have $y <_2 z$. Now there exists an element a making a triangle $a \to^{\mathbf{G}} x \to^{\mathbf{G}} z \to^{\mathbf{G}} a$. Now $y \leq_2 z \to^{\mathbf{G}} a$ implies $y \to^{\mathbf{T}_2} a$, while $a \to^{\mathbf{G}} x \leq_1 y$ implies $a \to^{\mathbf{T}_1} y$. Obviously, $a \neq y$ and we have that $a <_1 y$, $y <_2 a$. Now continuing to move through pairs of $\mathbf{G}$-triangles that share an edge, we find that every point w reachable through a sequence of such triangles satisfies $w \neq y$, $w <_1 y$, $y <_2 w$. This is a contradiction, because y is reachable. We conclude that $x \to^{\mathbf{T}_2} y$.

Case 3: $x <_1 z \to^{\mathbf{G}} y$. Here we find that either $x <_2 y$ or else $y <_2 x$ and $z <_2 x$. This case yields to essentially the same proof as in case 2.

Combining all cases, we find that $x \prec_1 y$ implies $x <_2 y$. Thus it follows by transitivity of $<_2$ that $x <_1 y$ implies $x <_2 y$. Since $<_1$ and $<_2$ have the same pairs of incomparable elements, we conclude that $<_1$ is identical with $<_2$, giving that $\mathbf{T}_1 = \mathbf{T}_2$. $\square$

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